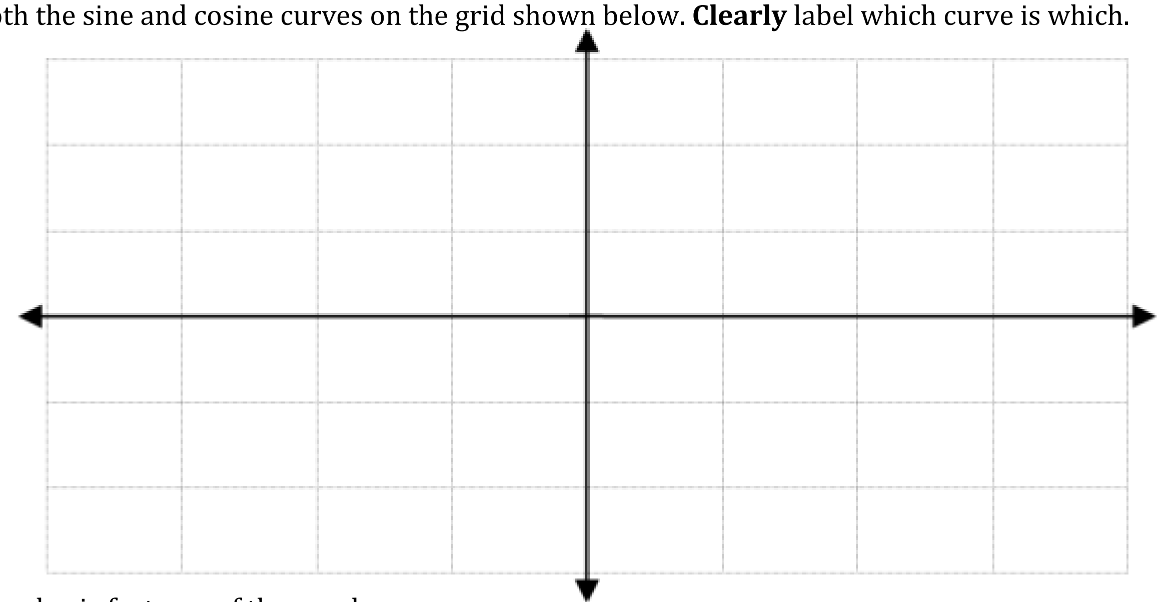
Chapter 11B: Trigonometric Graphing
Topic 1: Basic Graphs of Sine and Cosine

The graphs of sine and cosine functions are smooth, continuous, and cyclical. They originate from the "unwinding" of the unit circle. The sine and cosine functions can be easily graphed by considering their values at the quadrantal angles, those that are integer multiples of 90° or $\frac{\pi}{2}$ radians. Because of trig's use in future studies, we will do all graphing in RADIANS (even if we're thinking degrees in our work)

***Exercise* #1:** Consider the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is an angle in radians. By using the unit circle, fill out the table below for selected quadrantal angles.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
(degrees)									
$\sin x$									
$\cos x$									

Graph both the sine and cosine curves on the grid shown below. **Clearly** label which curve is which.



Summarize basic features of the graphs:

	$y = \sin x$	$y = \cos x$
y-intercept		
Symmetry		
Domain		
Range		
Period: Total measure of one **full** cycle		

AMPLITUDE:

Now we would like to explore the effect of changing the coefficient of the trigonometric function. In essence we would like to look at the graphs of functions of the forms:

$$\boldsymbol{y = A \sin x} \quad \text{and} \quad \boldsymbol{y = A \cos x}$$

The absolute value of the coefficient to the function, A, is called the **amplitude**. It indicates how **'loud'** the graph is. A positive vs. negative value indicates the starting **direction** of the graph.

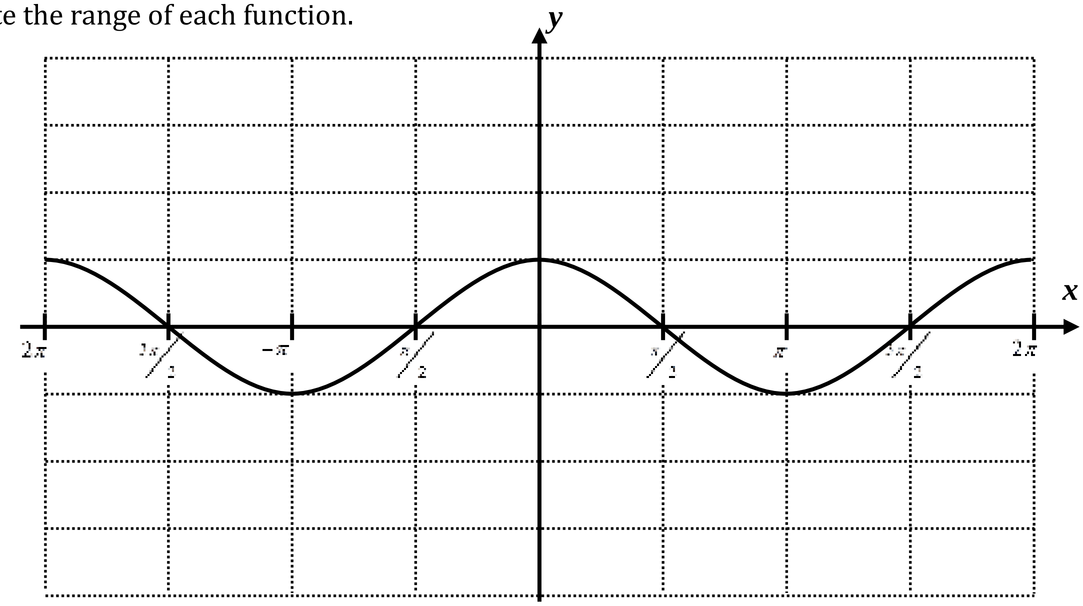
In this form, we can easily identify the range of the function to be "A" in either direction

Range: $-|A| \leq y \leq |A|$

***Exercise* #2:** The grid below shows the graph of $y = \cos x$. The leading coefficient (positive/negative, and its value) to help you graph. State the range of each function.

$y = 3\cos(x)$

$y = -4\cos(x)$

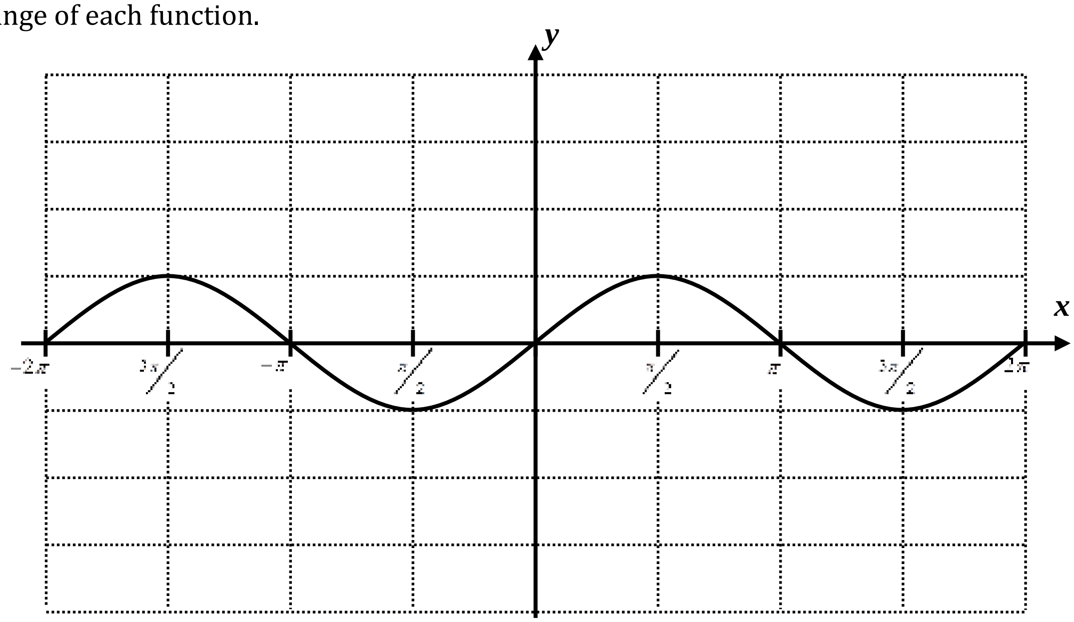


***Exercise* #3:** The basic sine function is graphed below. The leading coefficient (positive/negative, and its value) to help you graph. State the range of each function.

$y = 2\sin(x)$

$y = -3\sin(x)$

$y = -\dfrac{1}{2}\sin(x)$



MIDLINE:

For all of the graphs above, our midline was the x-axis. This r3eprsents the horizontal line which cuts through the center of the curve. When vertical shifts are introduced, the midline will change by the value being added or subtracted from the function, C.

$$\boldsymbol{y = A \sin x + C} \quad \text{and} \quad \boldsymbol{y = A \cos x + C}$$

To graph equations of this type, we start by considering how the graph has shifted vertically, then count the amplitude up and down from our midline. When the midline is anywhere other than the x-axis, it should be graphed (dotted) and labeled.

The range of the sine and cosine graph will change when a vertical shift is introduced. In positive terms of A & C:

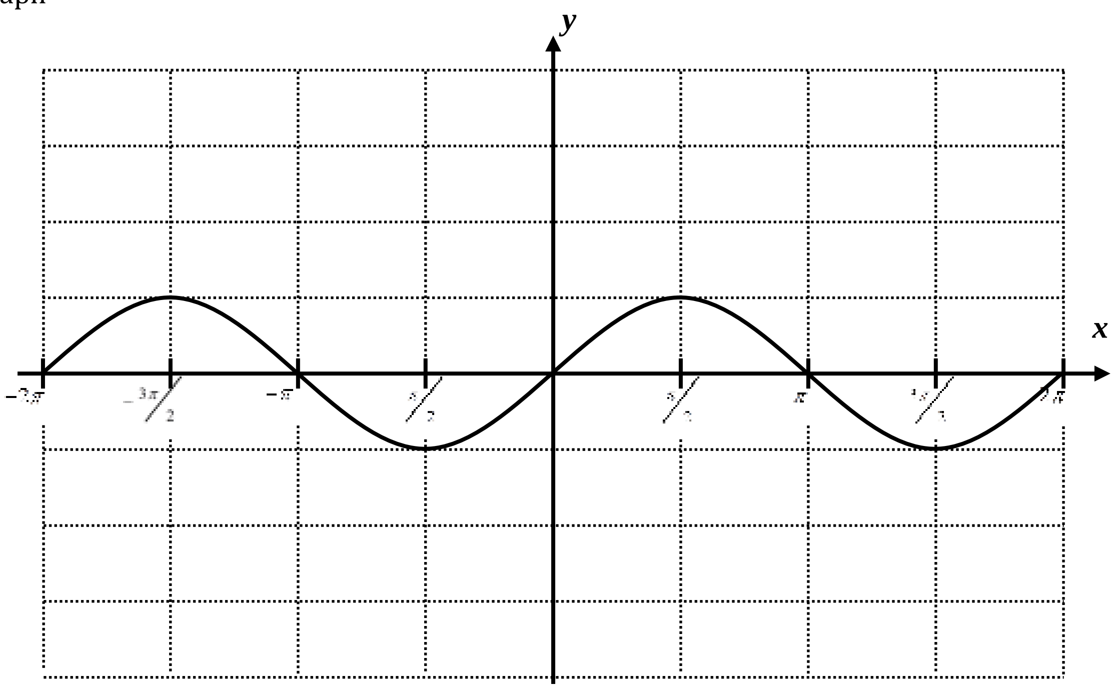
Range: $C - A \le y \le C + A$

***Exercise* #4:** Consider the function $f(x) = \sin(x) + 3$.

(a) How would the graph of $y = \sin(x)$ be shifted to produce the graph of f(x)?

(b) On the grid below is the basic sine curve, $y = \sin(x)$. On the same grid, sketch the graph of f(x). Start by dotting & labeling a midline and then counting amplitude.

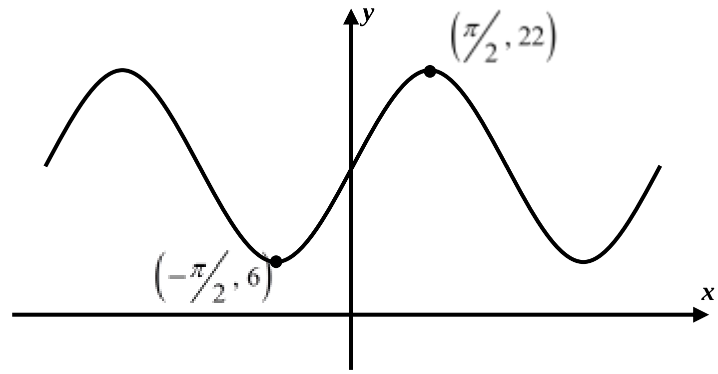
(c) State the range of your graph



***Exercise* #5:** Determine the range of each of the following trigonometric functions.

(a) $y = 7\sin(x) + 4$ (b) $y = -5\cos(x) + 2$ (c) $y = 25\sin(x) + 35$

***Exercise* #6:** The graph shows a sinusoidal curve of the form $y = A\sin(x) + C$. Find the values of C and A.



Using your calculator to graph trig functions

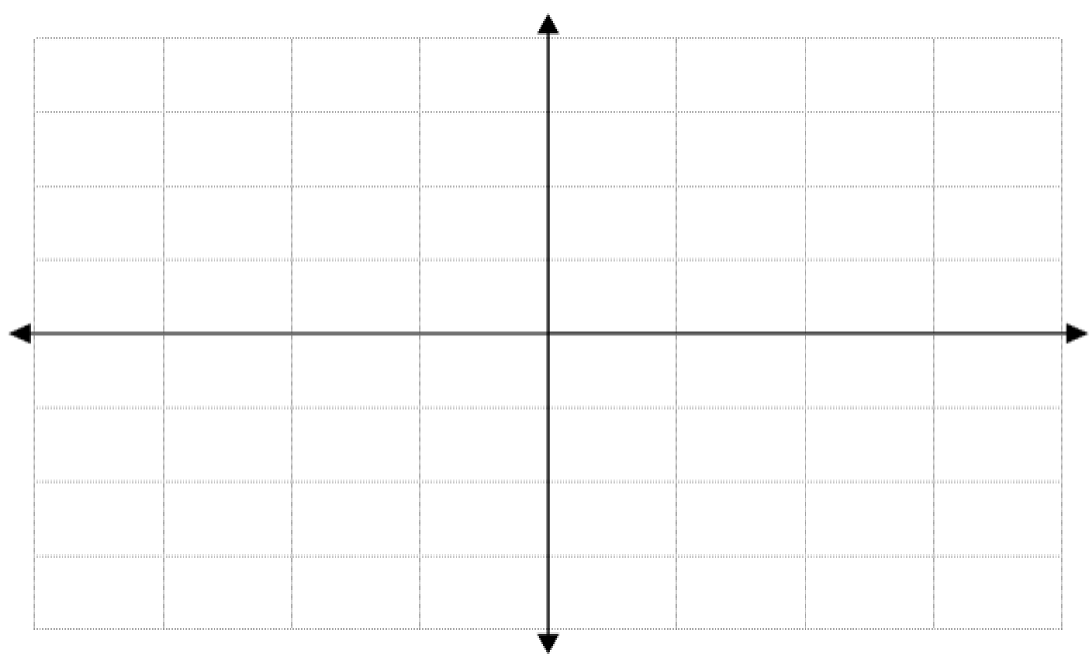
Due to its cyclical nature, it is very easy to graph trig functions without the use of a calculator, however sometimes it is quicker to use the calculator for help.

- Be sure you are in degree mode
- Carefully enter the function into "Y = "
- Set your table to "ASK" by choosing TBLSET (2^{nd} Window) and changing your Independent variable to "Ask"
- Enter your values (in degrees) in your table & graph the given points.

***Exercise* #7:** Given the function $y = 2\cos(x) + 1$.

(a) Graph the function. Remember, a fully complete graph is labeled, including a midline.

(b) State the range of your graph.

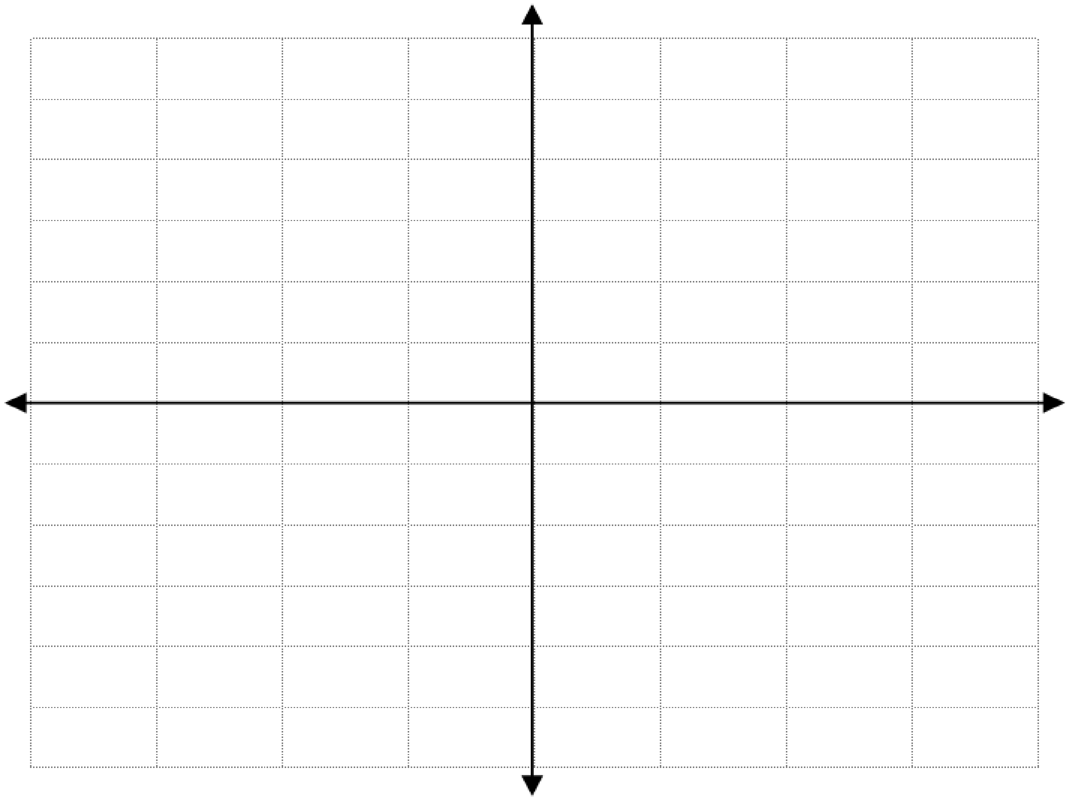


Topic 1 Homework: Basic Graphs of Sine and Cosine

1. On the grid below, sketch the graphs of each of the following equations based on the basic sine function. State the range of each

$y = 3\sin(x)$

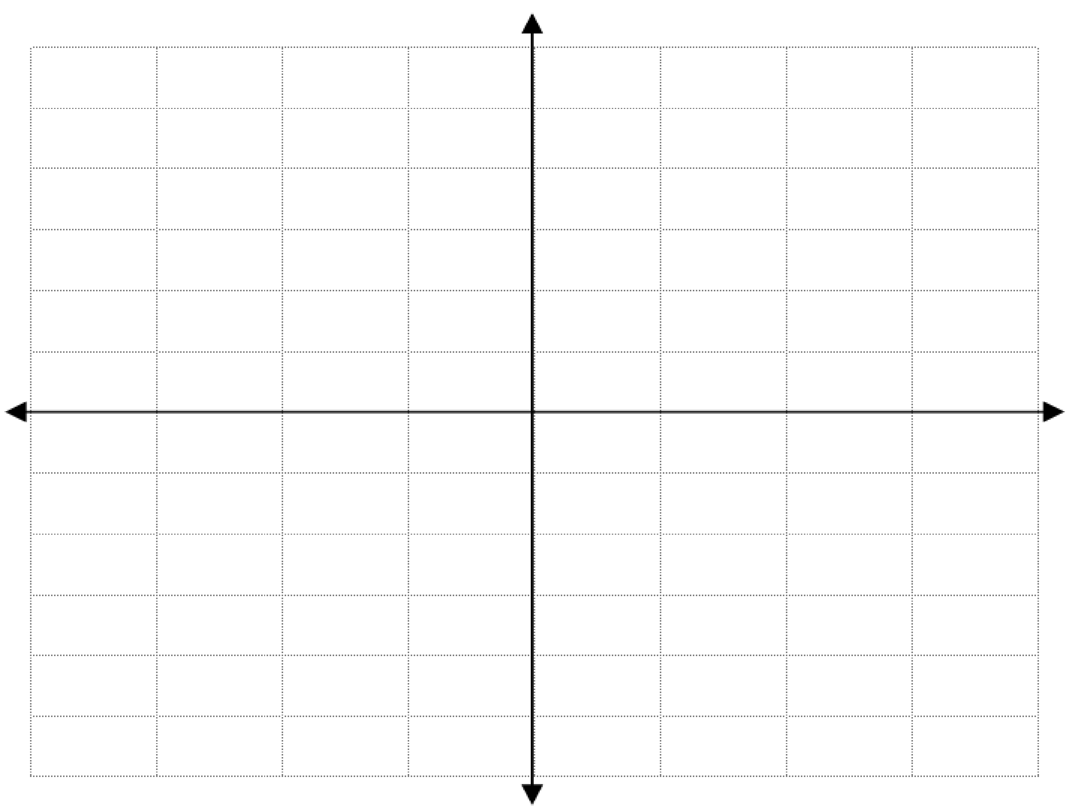
$y = -5\sin(x)$



2. On the grid below, sketch the graphs of each of the following equations based on the basic cosine function. State the range of each.

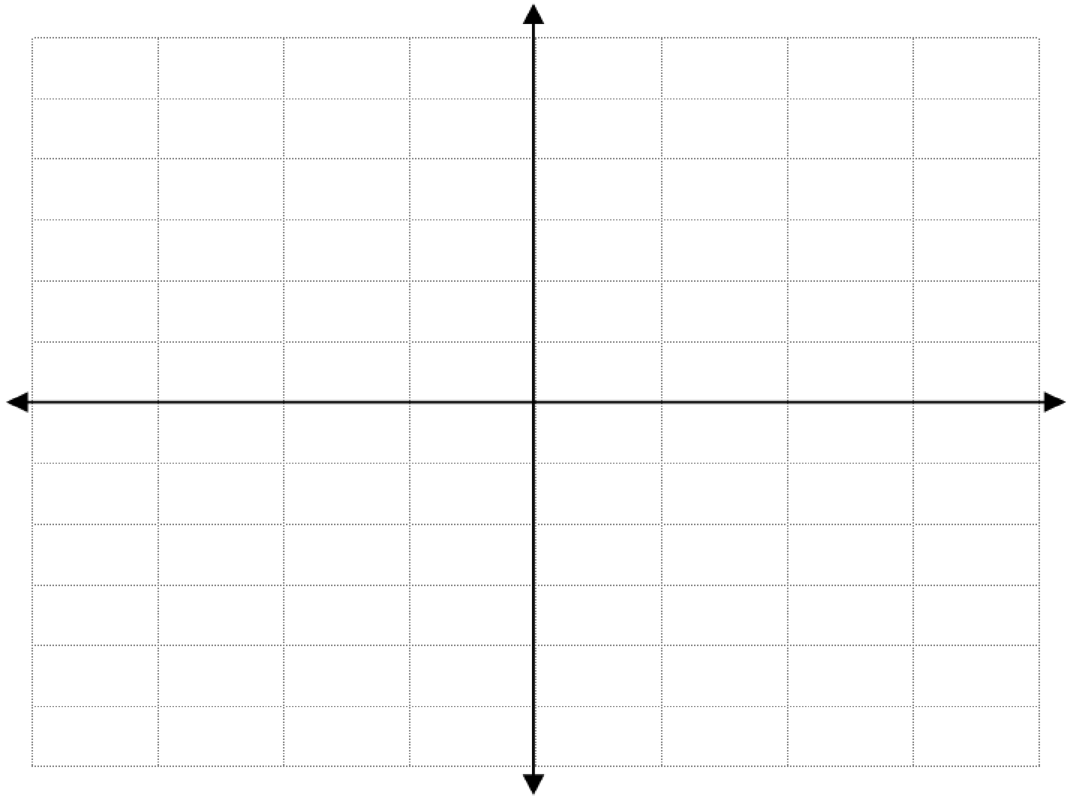
$y = 4\cos(x)$

$y = -3\cos(x)$



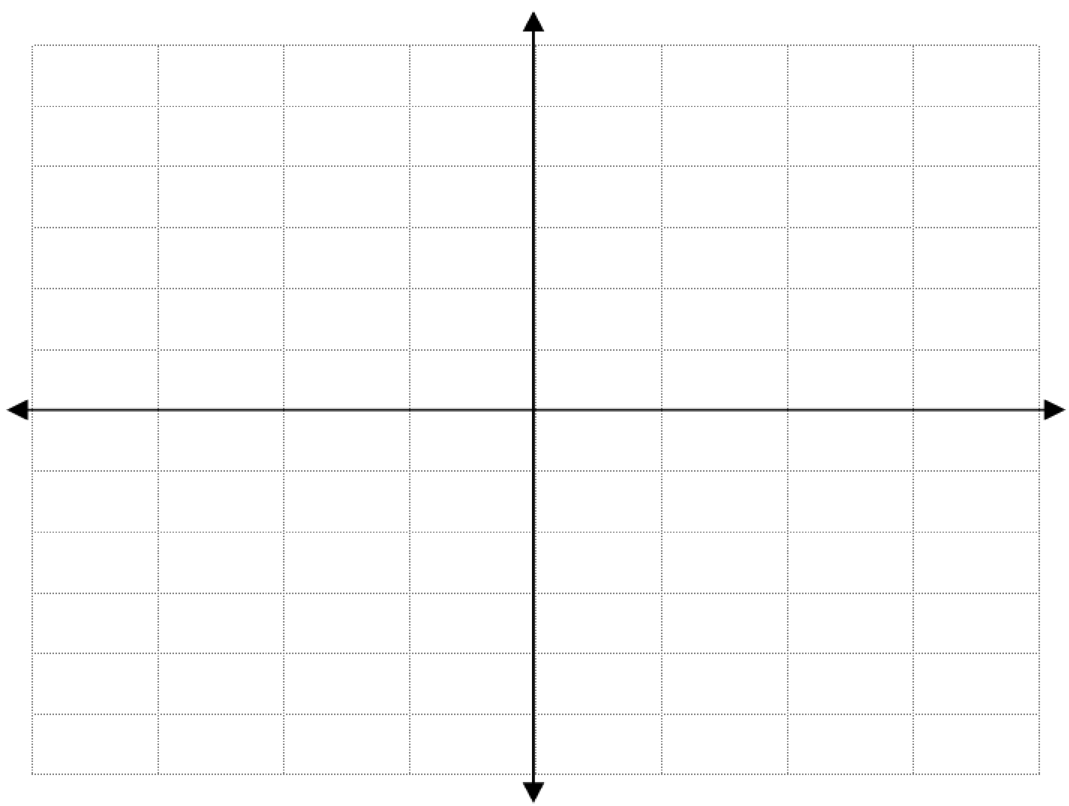
3. Graph and fully label the curve below. State the range.

$y = 4\sin(x) + 2$



4. Graph and fully label the curve below. State the range.

$y = -\cos(x) - 4$



5. Which of the following represents the *range* of the trigonometric function $y = 7\sin(x)$?

(1) $(-7, 7)$ (3) $[0, 7)$

(2) $[-7, 7]$ (4) $(-7, 7]$

6. Which of the following is the period of $y = \cos(x)$?

(1) π (3) 2π

(2) 2 (4) $\frac{3\pi}{2}$

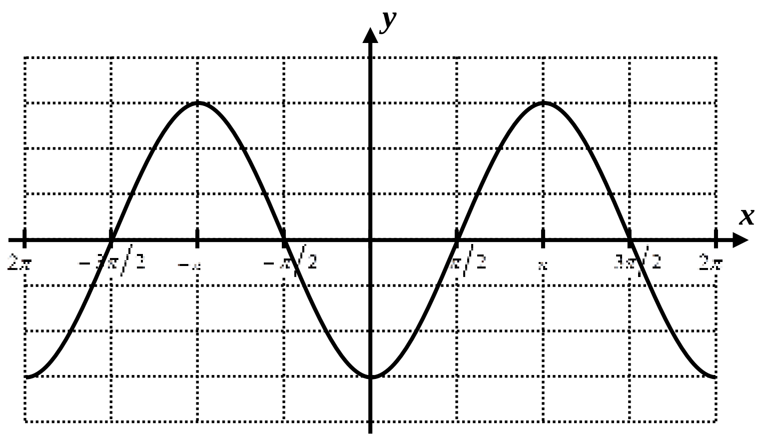
7. Which of the following equations describes the graph shown below?

(1) $y = 3\cos(x)$

(2) $y = -3\cos(x)$

(3) $y = 3\sin(x)$

(4) $y = -3\sin(x)$



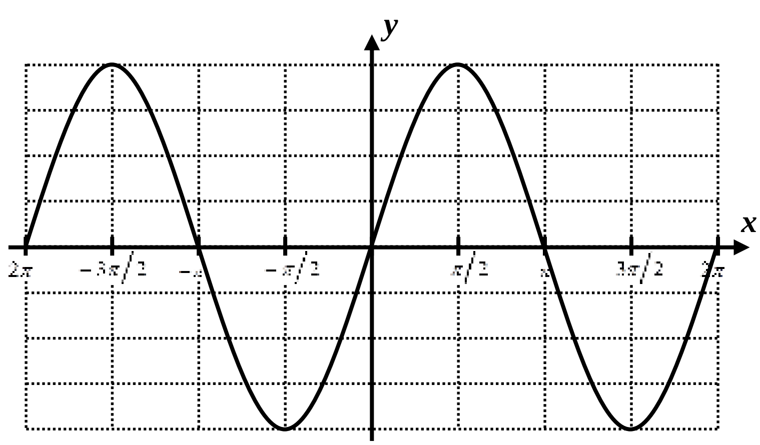
8. Which of the following equations represents the periodic curve shown below?

(1) $y = 4\cos(x)$

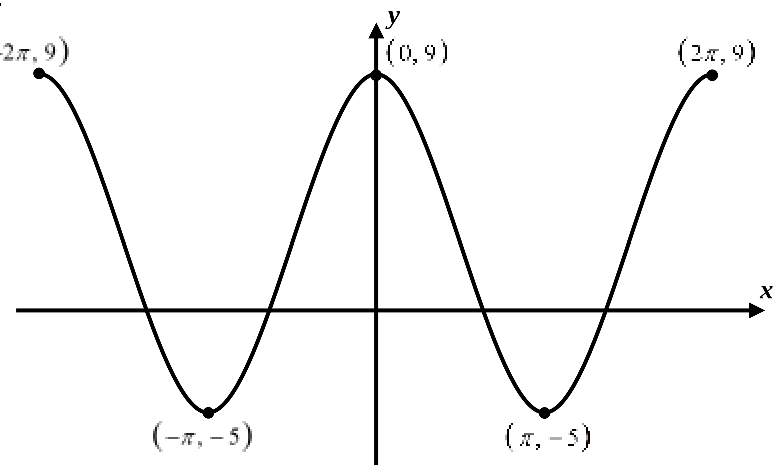
(2) $y = -4\cos(x)$

(3) $y = 4\sin(x)$

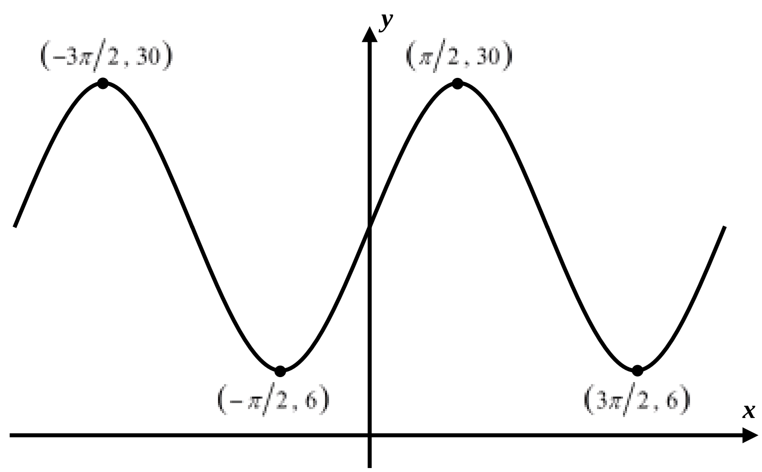
(4) $y = -4\sin(x)$



9. The following graph can be described using an equation of the form $y = A\cos(x) + C$. Determine the values of C and A. Show how you arrived at your answers.



10. The following graph can be described using an equation of the form $y = A\sin(x) + C$. Determine the values of C and A. Show how you arrived at your answers.



11. State the range of each of the following sinusoidal functions in interval form.

(a) $y = 10\sin(x) - 3$ (b) $y = -8\cos(x) + 2$ (c) $y = 22\sin(x) + 30$

12. When graphed, the line $y = 14$ would not intersect the graph of which of the following functions?

(1) $y = 5\cos(x) + 9$ (3) $y = 2\sin(x) + 15$

(2) $y = -6\cos(x) + 10$ (4) $y = 3\sin(x) + 20$

Chapter 11B: Trigonometric Graphing
Topic 2: Frequency & Period of the Sinusoidal Graph

A final transformation will allow us to horizontally stretch and compress sinusoidal graphs. It is important to be able to do this, especially when modeling real-world phenomena, because most **real life periodic functions** do not have a period of 2π. The first exercise will illustrate the pattern.

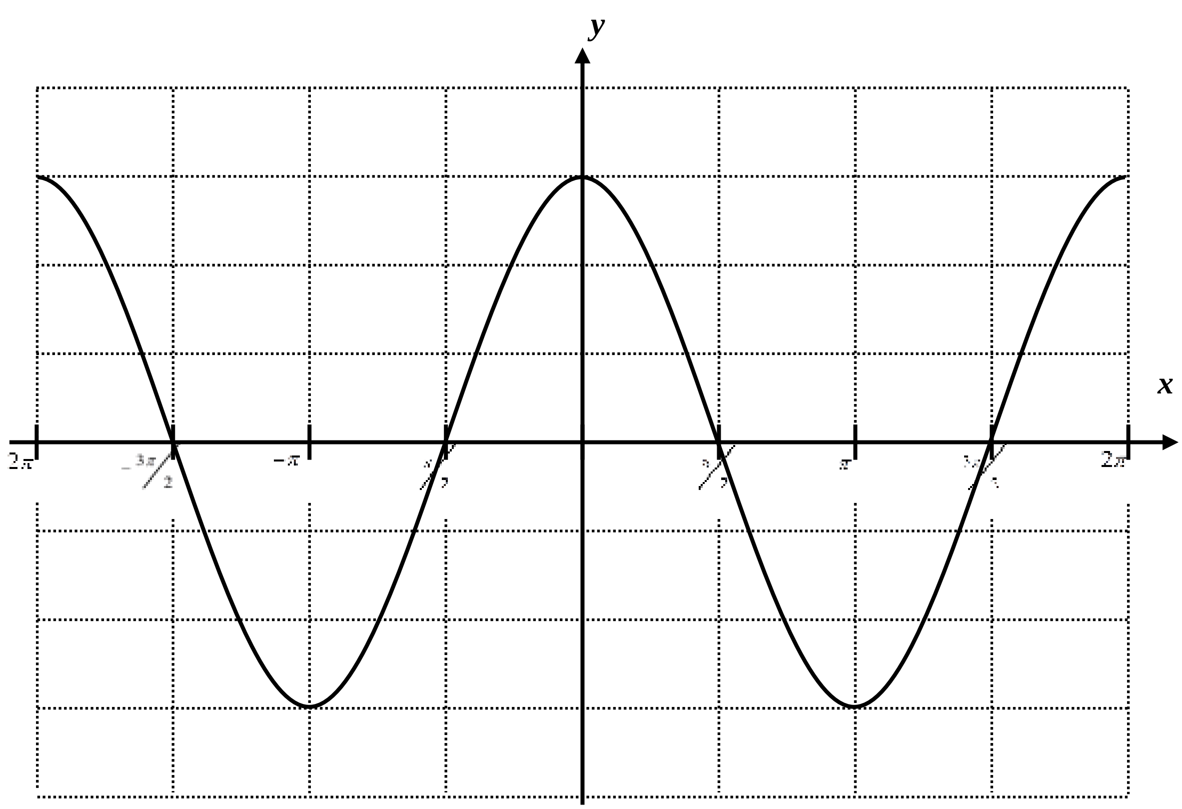
Exercise #1: On the grid below is a graph of the function $y = 3\cos(x)$.

(a) Using your calculator, sketch the graph of $y = 3\cos(2x)$ on the same axes.

(b) How many full cycles or periods of this function now fit within 2π radians?

(c) Using your calculator, sketch the graph of $y = 3\cos\left(\frac{1}{2}x\right)$ on the same axes.

(d) How many full cycles or periods of this function now fit within 2π radians?



The **period**, P, of a sinusoidal function is defined as **the minimum horizontal shift needed for the function to repeat its fundamental pattern**. The period for the basic sinusoidal graphs is 2π.

The period of the function depends on the coefficient B in the general equations. This coefficient, B, is known as the **frequency**.

$$y = A\sin(Bx) \quad \text{and} \quad y = A\cos(Bx)$$

<u>**Frequency**</u>: <u>**Period**</u>:

***Exercise* #2:** Consider the graphs from *Exercise* #1. For each below, state the frequency and period.

(a) $y = 3\cos(x)$

Frequency, $B =$

Period, $P =$

(b) $y = 3\cos(2x)$

Frequency, $B =$

Period, $P =$

(c) $y = 3\cos\left(\frac{1}{2}x\right)$

Frequency, $B =$

Period, $P =$

***Exercise* #3:** Determine the frequency & period of each of the following sinusoidal functions in exact terms.

(a) $y = 6\sin(4x)$

(b) $y = 8\cos\left(\frac{\pi}{3}x\right)$

(c) $y = -12\sin\left(\frac{2}{3}x\right)$

***Exercise* #4:** The heights of the tides can be described using a sinusoidal model of the form $y = A\cos(Bx) + C$. If high tides are separated by 24 hours, which of the following gives the frequency, B, of the curve?

(1) 12

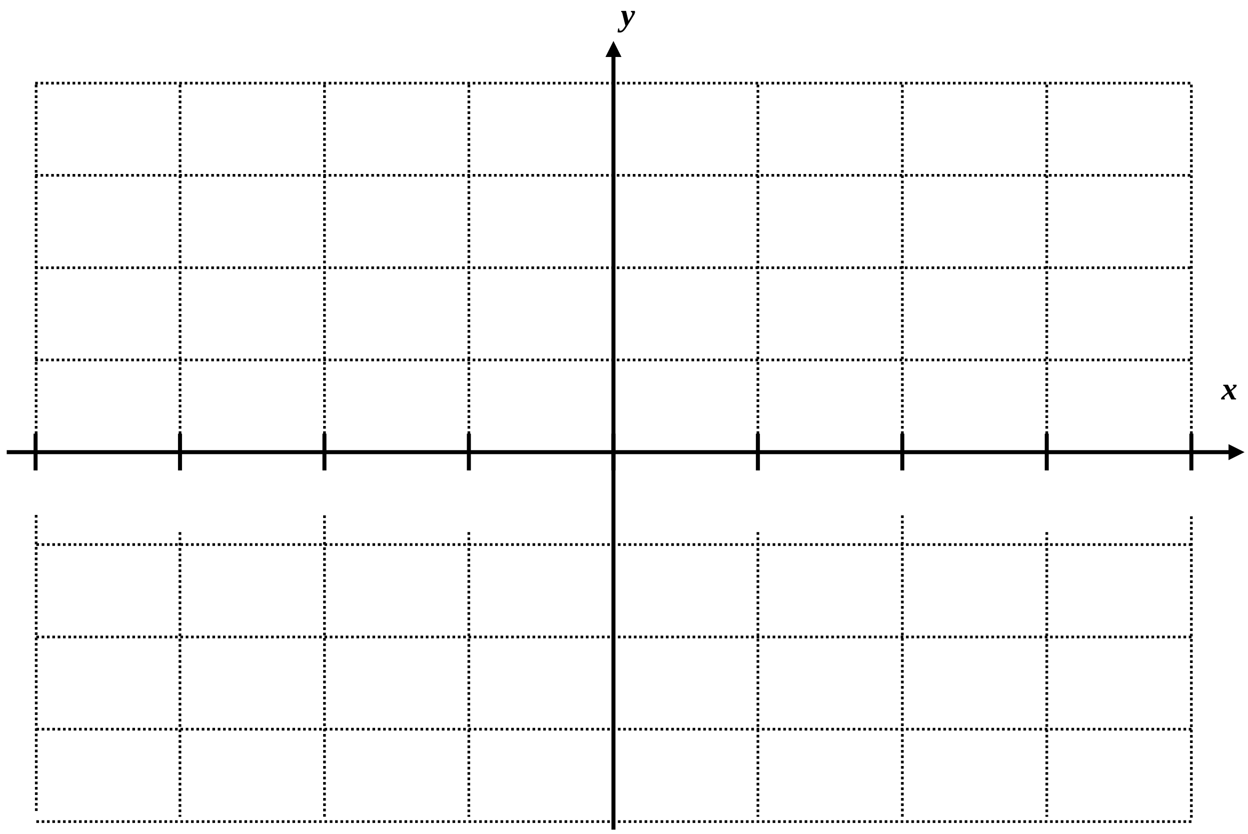
(2) $\frac{\pi}{24}$

(3) $\frac{\pi}{12}$

(4) $\frac{\pi}{6}$

***Exercise* #5:** Sketch the function $y = 2\sin(3x)$ on the grid below for one full period to the left and right of the y-axis. Start by looking at the quadrantial angle values in your table. Is counting by 90's sufficient?

A good rule to follow when B is different than 1: Count by $\frac{180}{2F}$ degrees.



Topic 2 Homework: Frequency & Period of the Sinusoidal Graph

1. For each of the following sinusoidal functions, determine its frequency & period in exact terms.

   (a) $y = 6\sin(10x)$ (b) $y = -2\cos(8x)$ (c) $y = 7\sin\left(\frac{1}{3}x\right)$

   (d) $y = \frac{2}{3}\cos\left(\frac{4}{3}x\right)$ (e) $y = 8\sin(0.25x)$ (f) $y = 2.5\cos(0.4x)$

2. If the period of a sinusoidal function is equal to 18, which of the following gives its frequency?

   (1) $\frac{\pi}{9}$ (3) $\frac{\pi}{18}$

   (2) 18π (4) 6π

3. It is known for that a particular sine curve repeats its fundamental pattern after every $\frac{2\pi}{7}$ units along the x-axis. Which of the following is the frequency of this curve?

   (1) $\frac{2}{7}$ (3) $\frac{7}{2}$

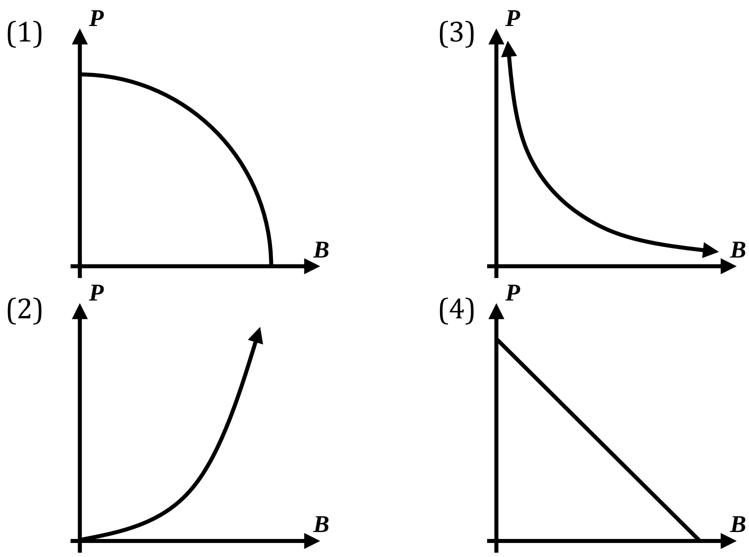
   (2) 7 (4) 14

4. When the period of a sine function doubles, the frequency

   (1) doubles. (3) is halved.

   (2) increases by 2. (4) decreases by 2.

5. Which of the following graphs shows the relationship between the frequency, B, and the period, P, of a sinusoidal graph? Experiment on your calculator. Graph the expression $P = \frac{2\pi}{B}$.

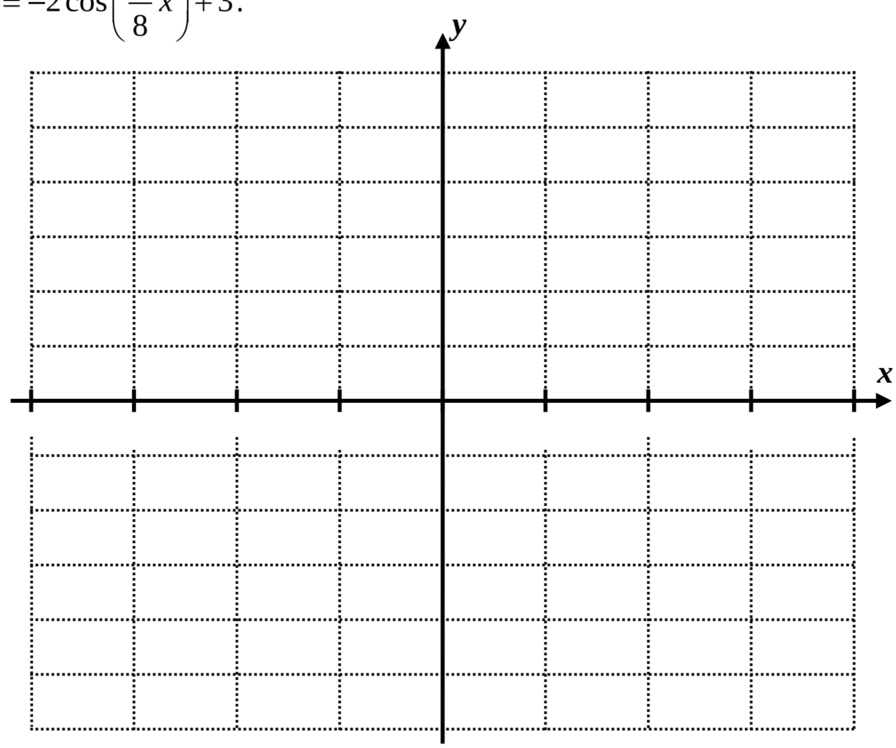


7. Consider the curve whose equation is $y = -2\cos\left(\frac{\pi}{8}x\right) + 3$.

(a) Determine the exact period of this sinusoidal function.

(b) What is the amplitude of this sinusoidal function?

(c) What is the midline value of this sinusoidal function?



(d) Sketch the function on the axes for a full period on both sides of the y-axis. Label the scale on your x-axis.

(e) What is the range of your graph?

Chapter 11B: Trigonometric Graphing
Topic 3: Sinusoidal Modeling

The sine and cosine functions can be used to model a variety of real-world phenomena that are periodic, that is, they repeat in predictable patterns. The key to constructing or interpreting a sinusoidal model is understanding the physical meanings of the coefficients we've explored in the last three lessons.

SINUSOIDAL MODEL COEFFICIENTS

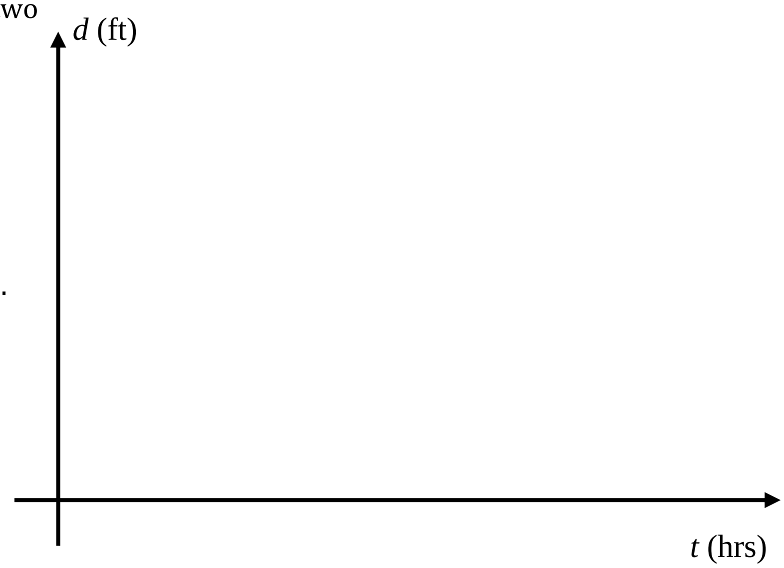
For $y = A\sin(Bx) + C$ and $y = A\cos(Bx) + C$

$\lvert A \rvert$	the **amplitude** or distance the sinusoidal model rises and falls above its midline
C	the **midline** or average *y*-value of the sinusoidal model
B	the **frequency** of the sinusoidal model – related to the **period**, *P*, by the equation $P = \frac{2\pi}{B}$
P	the **period** of the sinusoidal model – the minimum distance along the *x*-axis for the cycle to repeat

***Exercise* #1:** The tides in a particular bay can be modeled with an equation of the form $d = A\cos(Bt) + C$, where *t* represents the number of hours since high-tide and *d* represents the depth of water in the bay. The maximum depth of water is 36 feet, the minimum depth is 22 feet and high-tide is hit every 12 hours.

(a) On the axes, sketch a graph of this scenario for two full periods. Label the points on this curve that represent high and low tide.

(b) Determine the values of *A*, *B*, and *C* in the model. Verify your answers and sketch are correct on your calculator.



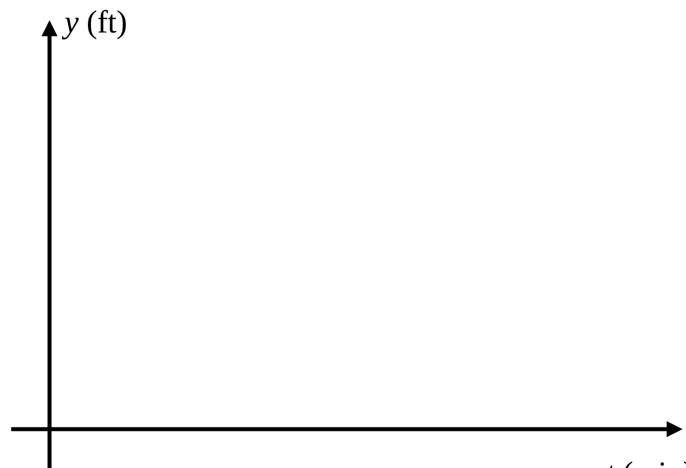
(c) Tanker boats cannot be in the bay when the depth of water is less or equal to 25 feet. Set up an inequality and solve it graphically to determine all points in time, *t*, on the interval $0 \le t \le 24$ when tankers cannot be in the bay. Round all times to the nearest *tenth* of an hour.

***Exercise* #2:** The height of a yo-yo above the ground can be well modeled using the equation $h = 1.75\cos(\pi t) + 2.25$, where h represents the height of the yo-yo in feet above the ground and t represents time in seconds since the yo-yo was first dropped from its maximum height.

(a) Determine the maximum and minimum heights that the yo-yo reaches above the ground. Show the calculations that lead to your answers.

(b) How much time does it take for the yo-yo to return to the maximum height for the first time?

***Exercise* #3:** A Ferris wheel is constructed such that a person gets on the wheel at its lowest point, five feet above the ground, and reaches its highest point at 130 feet above the ground. The amount of time it takes to complete one full rotation is equal to 8 minutes. A person's vertical position, *y*, can be modeled as a function of time in minutes since they boarded, *t*, by the equation $y = A\cos(Bt) + C$. Sketch a graph of a person's vertical position for one cycle and then determine the values of *A*, *B*, and *C*. Show the work needed to arrive at your answers.



***Exercise* #4:** The possible hours of daylight in a given day is a function of the day of the year. In Poughkeepsie, New York, the minimum hours of daylight (occurring on the Winter solstice) is equal to 9 hours and the maximum hours of daylight (occurring on the Summer solstice) is equal to 15 hours. If the hours of daylight can be modeled using a sinusoidal equation, what is the equation's ampltitude?

(1) 6 (2) 12 (3) 3 (4) 4

Topic 3 Homework: Sinusoidal Modeling

1. A ball is attached to a spring, which is stretched and then let go. The height of the ball is given by the sinusoidal equation $y = -3.5\cos\left(\frac{4\pi}{5}t\right) + 5$, where y is the height above the ground in feet and t is the number of seconds since the ball was released.

   (a) At what height was the ball released at? Show the calculation that leads to your answer.

   (b) What is the maximum height the ball reaches?

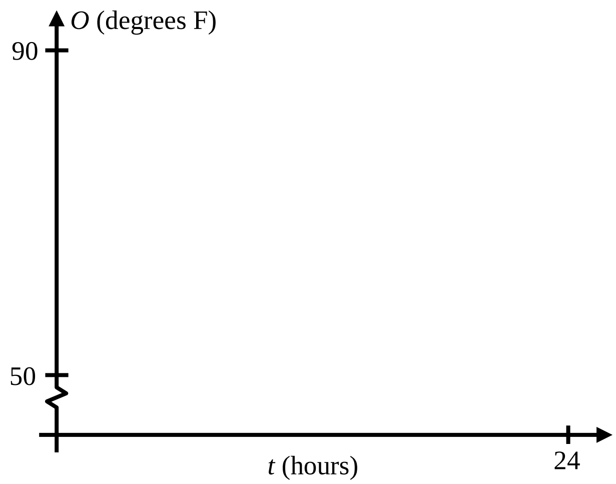
   (c) How many seconds does it take the ball to return to its original position?

   (d) Draw a rough sketch of one complete period of this curve below. Label maximum and minimum points.

2. An athlete was having her blood pressure monitored during a workout. Doctors found that her maximum blood pressure, known as systolic, was 110 and her minimum blood pressure, known as diastolic, was 70. If each heartbeat cycle takes 0.75 seconds, then determine a sinusoidal model, in the form $y = A\sin(Bt) + C$, for her blood pressure as a function of time t in seconds. Show the calculations that lead to your answer.

3. On a standard summer day in upstate New York, the temperature outside can be modeled using the sinusoidal equation $O(t) = 11\cos\left(\frac{\pi}{12}t\right) + 71$, where *t* represents the number of hours since the peak temperature for the day.

(a) Sketch a graph of this function on the axes below for one day.



(b) For $0 \le t \le 24$, graphically determine all points in time when the outside temperature is equal to 75 degrees. Round your answers to the nearest tenth of an hour.

4. The percentage of the moon's surface that is visible to a person standing on the Earth varies with the time since the moon was full. The moon passes through a full cycle in 28 days, from full moon to full moon. The maximum percentage of the moon's surface that is visible is 50%. Determine an equation, in the form $P = A\cos(Bt) + C$ for the percentage of the surface that is visible, *P*, as a function of the number of days, *t*, since the moon was full. Show the work that leads to the values of *A*, *B*, and *C*.

5. Evie is on a swing thinking about trigonometry (no seriously!). She realizes that her height above the ground is a periodic function of time that can be modeled using $h = 3\cos\left(\frac{\pi}{2}t\right) + 5$, where *t* represents time in seconds. Which of the following is the range of Evie's heights?

(1) $2 \le h \le 8$ (3) $3 \le h \le 5$

(2) $4 \le h \le 8$ (4) $2 \le h \le 5$

Chapter 11B: Trigonometric Graphing
Topic 4: The Reciprocal Trig Functions

We have now seen three primary trigonometric functions, the sine, cosine, and tangent functions. Each of these can be defined in terms of either **ratios of the sides of a right** triangle or **the unit circle**. For each of these functions, though, there exists what is known as a **reciprocal function**. Their definitions are shown below.

THE OTHER THREE TRIGONOMETRIC FUNCTIONS

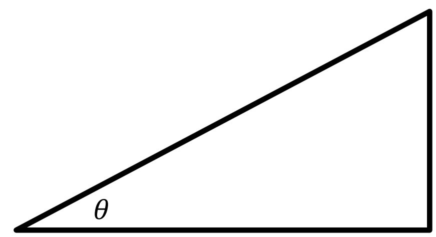
1. SECANT: $\sec(\theta) = \dfrac{1}{\cos(\theta)}$

2. **COSECANT:** $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

3. **COTANGENT:** $\cot(\theta) = \dfrac{1}{\tan(\theta)}$ or equivalently $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

Considering our reciprocal functions in a few ways:

- **SOHCAHTOA**



- **EXACT VALUES**

$\boldsymbol{\theta}$	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**	**360°**
sin $\boldsymbol{\theta}$								
cos $\boldsymbol{\theta}$								
tan $\boldsymbol{\theta}$								

$\boldsymbol{\theta}$	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**	**360°**
csc $\boldsymbol{\theta}$								
sec $\boldsymbol{\theta}$								
cot $\boldsymbol{\theta}$								

***Exercise* #1:** Determine the sign of each of the following trigonometric functions in the quadrant specified.

(a) $\cot\theta$ for θ in quad. II

(b) $\sec\theta$ for θ in quad. IV

(c) $\csc\theta$ for θ in quad. III

***Exercise* #2:** If $\cot\theta < 0$ and $\sec\theta > 0$ then θ could be which of the following angles?

(1) $\theta = 48^\circ$ (2) $\theta = 310^\circ$

(3) $\theta = 122^\circ$ (4) $\theta = 225^\circ$

***Exercise* #3:** Considering your work with sine and cosine, evaluate each of the following. Express your answers in exact and simplest form.

(a) $\sec 120°$ (b) $\cot 150°$ (c) $\csc\frac{3\pi}{4}$

***Exercise* #4:** Which of the following is closest to the value of $\sec 52°$?

(1) 0.62 (2) 1.62 (3) 0.36 (4) 2.48

***Exercise* #5:** Which of the following values of x is *not* in the domain of $y = \csc x$?

That is: Which number will have a sine of zero, therefore making cosecant $\left(\frac{1}{\sin\theta}\right)$ *undefined?*

(1) $x = 180^{\circ}$ (2) $x = 60^{\circ}$ (3) $x = 90^{\circ}$ (4) $x = 135^{\circ}$

***Exercise* #6:** If θ is an angle whose terminal ray lies in the fourth quadrant and $\cos\theta = \frac{7}{25}$, then determine the exact value of $\csc\theta$.

***Exercise* #7:** If θ is an angle whose terminal ray lies in the second quadrant and $\tan\theta = \frac{-3}{4}$, then determine the exact value of $\sec\theta$.

Topic 4 Homework: The Reciprocal Trig Functions

1. Determine the value of each of the following in exact and simplest form (leave no complex fractions).

   (a) $\csc\left(30^\circ\right)$ (b) $\cot\left(90^\circ\right)$ (c) $\sec\left(180^\circ\right)$

   (d) $\cot\left(\frac{\pi}{3}\right)$ (e) $\csc\left(\frac{3\pi}{2}\right)$ (f) $\sec\left(\frac{5\pi}{4}\right)$

2. Use your calculator to determine the value of each of the following to the nearest *hundredth*.

   (a) $\cot\left(115^\circ\right)$ (b) $\sec\left(312^\circ\right)$ (c) $\csc\left(245^\circ\right)$

3. In simplest radical form, $\sec\left(135^\circ\right)$ is equal to

   (1) $-\frac{\sqrt{2}}{3}$ (3) $-\frac{\sqrt{2}}{2}$

   (2) $-\sqrt{2}$ (4) $-\frac{\sqrt{3}}{2}$

4. Which of the following is nearest to the value of $\cot\left(220^{\circ}\right)$?

(1) 1.19 (2) 3.17 (3) –2.74 (4) –0.85

5. For which of the following values of α is $\cot(\alpha)$ undefined?

(1) 60° (2) 90° (3) 180° (4) 135°

6. For which angle, β, below will $\sec(\beta)$ not exist?

(1) 30° (2) 45° (3) 180° (4) 90°

7. For the angle β it is known that $\csc(\beta)>0$ and $\sec(\beta)<0$. When drawn in standard position, the terminal ray of β lies in quadrant

(1) I (2) II (3) III (4) IV

8. The angle θ when drawn in standard position has its terminal ray in the second quadrant. If it is known that $\sin\theta=\frac{5}{13}$ then determine the values of all of the remaining trigonometric functions.

(a) $\cos\theta$

(b) $\tan\theta$

(c) $\sec\theta$

(d) $\csc\theta$

(e) $\cot\theta$