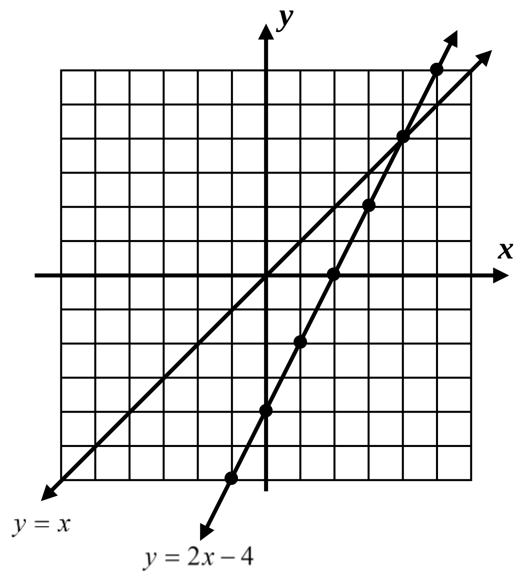
LESSON 5
INVERSE OF LINEAR FUNCTIONS

Recall that functions have inverses that are also functions if they are one-to-one. With the exception of horizontal lines, all linear functions are one-to-one and thus have inverses that are also functions. In this lesson we will investigate these inverses and how to find their equations.

***Exercise* #1:** On the grid below the linear function $y = 2x - 4$ is graphed along with the line $y = x$.

(a) How can you quickly tell that $y = 2x - 4$ is a one-to-one function?



(b) Graph the inverse of $y = 2x - 4$ on the same grid. Recall that this is easily done by switching the x and y coordinates of the original line.

(c) What can be said about the graphs of $y = 2x - 4$ and its inverse with respect to the line $y = x$?

(d) Find the equation of the inverse in .
b form.

(e) Find the equation of the inverse in $y = \frac{x+b}{a}$ $y = mx +$
form.

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As we can see from part (e) in *Exercise* #1, inverses of linear functions include the inverse operations of the original function but in reverse order. This gives rise to a simple method of finding the equation of any inverse. **Simply switch the *x* and *y* variables in the original equation and solve for *y*.**

____ ***Exercise* #2:** Which of the following represents the equation of the inverse of $y = 5x - 20$.

(1) $y = -\frac{1}{5}x + 20$

(2) $y = \frac{1}{5}x - 20$

(3) $y = \frac{1}{5}x - 4$

(4) $y = \frac{1}{5}x + 4$

Although this is a simple enough procedure, certain problems can lead to common errors when solving for *y*. Care should be taken with each algebraic step.

____ ***Exercise* #3:** Which of the following represents the inverse of the linear function $y = \frac{2}{3}x + 8$?

(1) $y = \frac{3}{2}x - 8$

(2) $y = \frac{3}{2}x - 12$

(3) $y = -\frac{3}{2}x + 8$

(4) $y = -\frac{3}{2}x + 12$

____ ***Exercise* #4:** What is the *y*-intercept of the inverse of $y = \frac{3}{5}x - 9$?

(1) $y = 15$

(2) $y = \frac{1}{9}$

(3) $y = 9$

(4) $y = -\frac{5}{3}$

Sometimes we are asked to work with linear functions in their point-slope form. The method of finding the inverse and plotting it, though, do not change just because the linear equation is written in a different form.

_____***Exercise* #5:** Which of the following would be an equation for the inverse of $y+6=4(x-2)$?

(1) $y-2=\frac{1}{4}(x+6)$ (3) $y-6=-4(x+2)$

(2) $y-2=-\frac{1}{4}(x+6)$ (4) $y+2=-4(x-6)$

_____***Exercise* #6:** Which of the following points lies on the graph of the inverse of $y-8=5(x+2)$? Explain your choice.

(1) $(8,-2)$ (3) $(-10,40)$

(2) $(-8,2)$ (4) $(-2,8)$

_____***Exercise* #7:** Which of the following linear functions would *not* have an inverse that is also a function? Explain how you made your choice.

(1) $y=x$ (3) $y=2$

(2) $2y=x$ (4) $y=5x-1$

LESSON 5
HOMEWORK

_____1. The graph of a function and its inverse are always symmetric across which of the following lines?

(1) $y = 0$ (3) $y = x$

(2) $x = 0$ (4) $y = 1$

_____2. Which of the following represents the inverse of the linear function $y = 3x - 24$?

(1) $y = \frac{1}{3}x + 8$ (3) $y = -\frac{1}{3}x + 24$

(2) $y = -\frac{1}{3}x - 8$ (4) $y = \frac{1}{3}x - \frac{1}{24}$

_____3. If the y-intercept of a linear function is 8, then we know which of the following about its inverse?

(1) Its y-intercept is -8.   (3) Its y-intercept is $\frac{1}{8}$.

(2) Its x-intercept is 8.   (4) Its x-intercept is -8.

_____4. If both were plotted, which of the following linear functions would be parallel to its inverse? Explain your thinking.

(1) $y = 2x$ (3) $y = 5x - 1$

(2) $y = \frac{2}{3}x - 4$ (4) $y = x + 6$

_____5. Which of the following represents the equation of the inverse of $y = \frac{4}{3x} + 24$?

(1) $y = -\frac{4}{3}x - 24$ (3) $y = \frac{3}{4}x - 18$

(2) $y = -\frac{3}{4}x + 18$ (4) $y = \frac{4}{3}x - 24$

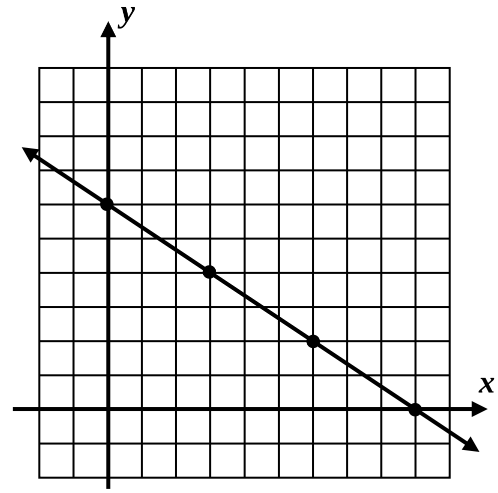
_____6. Which of the following points lies on the inverse of $y + 2 = 4(x - 1)$?

(1) $(2, -1)$ (3) $\left(\frac{1}{2}, 1\right)$

(2) $(-1, 2)$ (4) $(-2, 1)$

7. A linear function is graphed below. Answer the following questions based on this graph.

(a) Write the equation of this linear function in $y = mx + b$ form.



(b) Sketch a graph of the inverse of this function on the same grid.

(c) Write the equation of the inverse in $y = mx + b$ form.

(d) What is the intersection point of this line with its inverse?

8. A car traveling at a constant speed of 58 miles per hour has a distance of y-miles from Poughkeepsie, NY, given by the equation $y = 58x + 24$, where x represents the time in hours that the car has been traveling.

(a) Find the equation of the inverse of this linear function in $y = \frac{(x-a)}{b}$ form.

(b) Evaluate the function you found in part (a) for an input of x = 227.

(c) Give a physical interpretation of the answer you found in part (b). Consider what the input and output of the inverse represent in order to answer this question.

9. Given the general linear function $y = mx + b$, find an equation for its inverse in terms of m and b.

Answers to Lesson 5 - Homework

1) (3)

2) (1)

3) (2)

4) (4)

5) (3)

6) (4)

7) (a) $y = -\frac{2}{3}x + 6$

   (b) Graph a line by switching the x, and y-coordinates from the given line.

   (c) $y = -\frac{3}{2}x + 9$

   (d) $\left(\frac{18}{5}, \frac{18}{5}\right)$

8) (a) $y = \frac{x-24}{58}$ (b) $y = 3.5$

   (c) In the original function x represented the time and y represented the distance. For the inverse, these two must be switched, thus x represents the distance and y represents the time. So, the interpretation of (b) is that it takes 3.5 hours to reach a distance of 227 miles from Poughkeepsie.

9) $y = \frac{1}{m}x - \frac{b}{m}$ or $y = \frac{x-b}{m}$

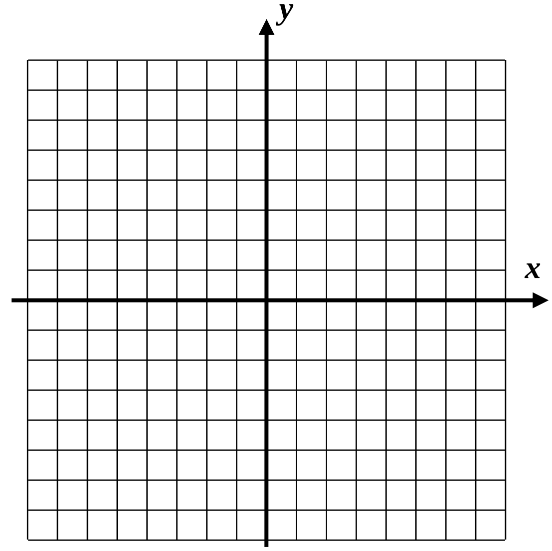
LESSON 6
PIECEWISE LINEAR FUNCTIONS

Functions expressed algebraically can sometimes be more complicated and involve different equations for different portions of their domains. These are known as piecewise functions (they come in pieces). If all of the pieces are linear, then they are known as piecewise linear functions.

Exercise #1: Consider the piecewise linear function given by the formula $\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x-3} & \boldsymbol{-3 \le x < 0} \\ \boldsymbol{\frac{1}{2}x+4} & \boldsymbol{0 \le x \le 4} \end{cases}$

(a) Create a table of values below and graph the function.

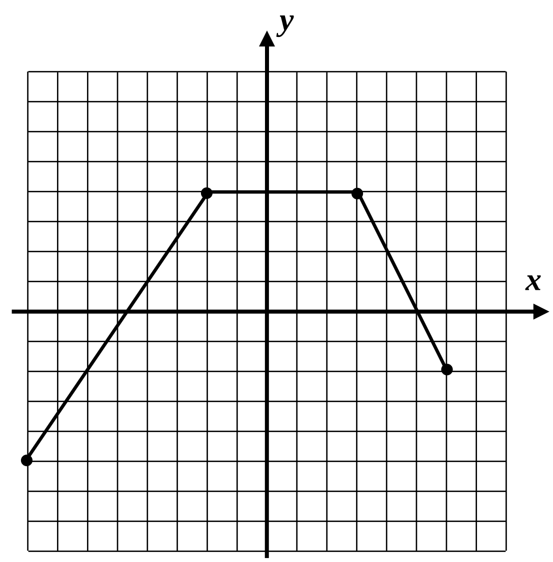
x	–3	–2	–1	0	1	2	3	4
$f(x)$								



(b) State the range of f using interval notation.

Not only should we be able to graph piecewise functions when we are given their equations, but we should also be able to translate the graphs of these functions into equations.

***Exercise* #2:** The function $f(x)$ is shown graphed below. Write a piecewise linear formula for the function. Be sure to specify both the formulas and the domain intervals over which they apply.



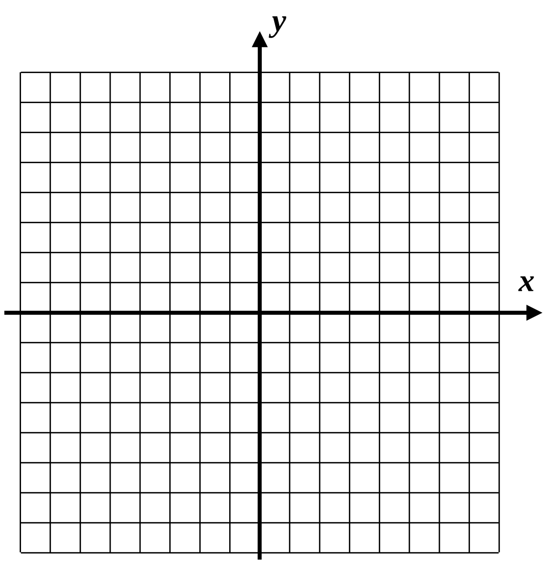
Piecewise equations can be challenging algebraically. Sometimes information that we find from them can be misleading or incorrect.

***Exercise* #3:** Consider the piecewise linear function $g(x) = \begin{cases} 5 - x & x < 2 \\ \frac{1}{2}x + 2 & x \geq 2 \end{cases}$

(a) Determine the y-intercept of this function algebraically. Why can a function have only one y-intercept?

(b) Find the x-intercepts of each individual linear equation.

(c) Graph the piecewise linear function below.



(d) Why does your graph contradict the answers you found in part (b)?

(e) How can you resolve the fact that the algebra seems to contradict your graphical evidence of x-intercepts?

***Exercise* #4:** For the piecewise linear function $f(x) = \begin{cases} -2x + 10 & x \leq 0 \\ 5x - 1 & x > 0 \end{cases}$, find all solutions to the equation $f(x) = 1$ algebraically.

LESSON 6
HOMEWORK

1. For $f(x) = \begin{cases} 5x - 3 & x < -2 \\ x + 8 & -2 \le x < 3 \\ \frac{1}{3}x + 7 & x \ge 3 \end{cases}$ answer the following questions.

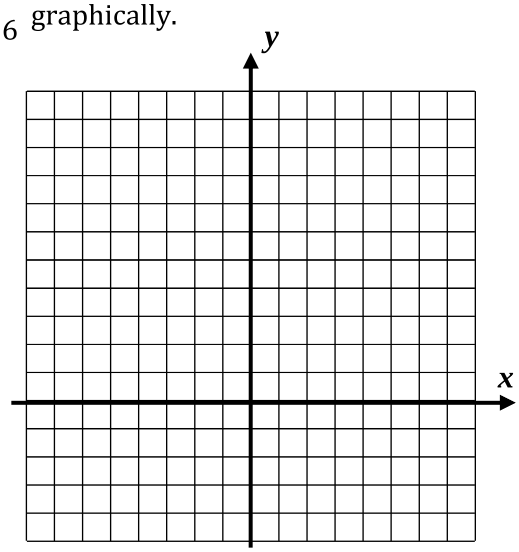
   (a) Evaluate each of the following by carefully applying the correct formula:

   (i) $f(2)$ (ii) $f(-4)$ (iii) $f(3)$ (iv) $f(0)$

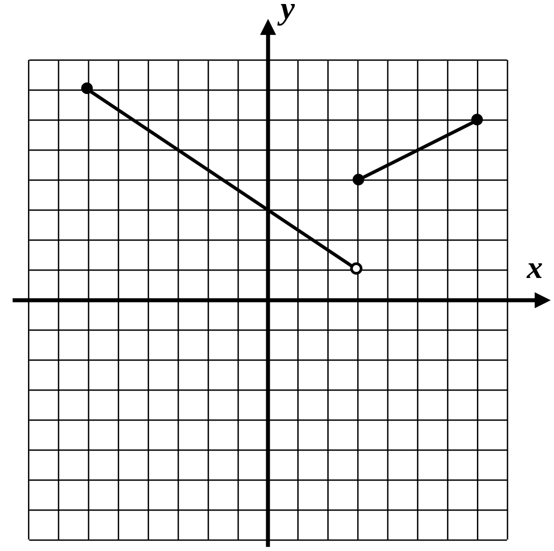
   (b) The three linear equations have y-intercepts of $-3, 8,$ *and* 7 respectively. Yet, a function can have only one y-intercept. Which of these is the y-intercept of this function? Explain how you made your choice.

   (c) Calculate the average rate of change of f over the interval $-3 \le x \le 9$. Show the calculations that lead to your answer.

2. Determine the range of the function $g(\boldsymbol{x}) = \begin{cases} x + 4 & -2 \le x < 2 \\ -\frac{3}{2}x + 9 & 2 < x \le 6 \end{cases}$ graphically.

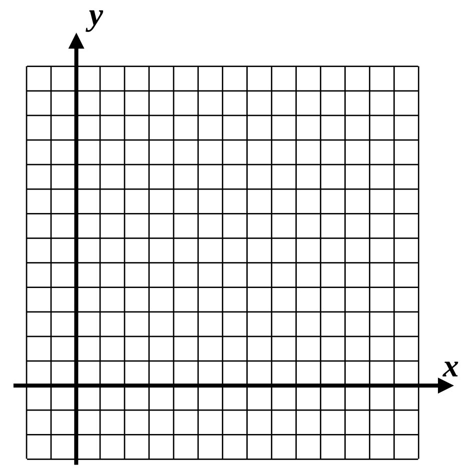


3. Determine a piecewise linear equation for the function $f(x)$ shown below. Be sure to specify not only the equations, but also the domain intervals over which they apply.



4. Step functions are piecewise functions that are constants (horizontal lines) over each part of their domains. Graph the following step function.

$$f(x)\begin{cases} -2 & 0 \le x < 3 \\ 3 & 3 \le x < 5 \\ 7 & 5 \le x < 10 \\ 5 & 10 \le x \le 12 \end{cases}$$



5. Find all x-intercepts of the function $g(x) = \begin{cases} 2x+8 & -5 \le x < -1 \\ -\frac{1}{2}x - 4 & -1 \le x < 1 \\ -4x+10 & 1 \le x \le 4 \end{cases}$ algebraically. Justify your work by showing your algebra. Be sure to check your answers versus the domain intervals to make sure each solution is valid.

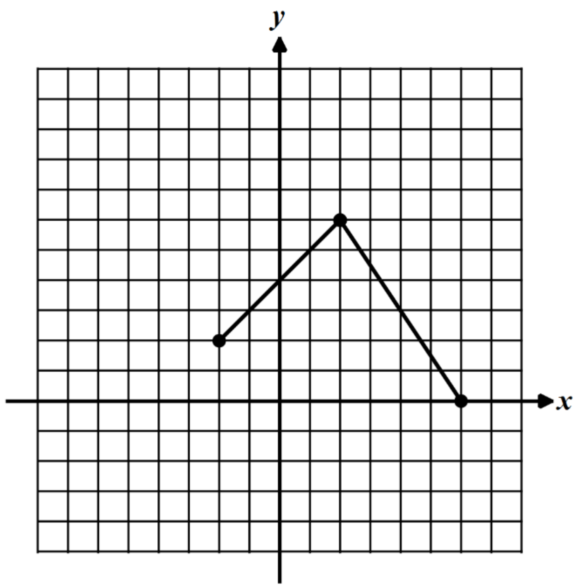
Answers to Lesson 6 – Homework

1) (a) (i) 10 (ii) -23 (iii) 8 (iv) 8

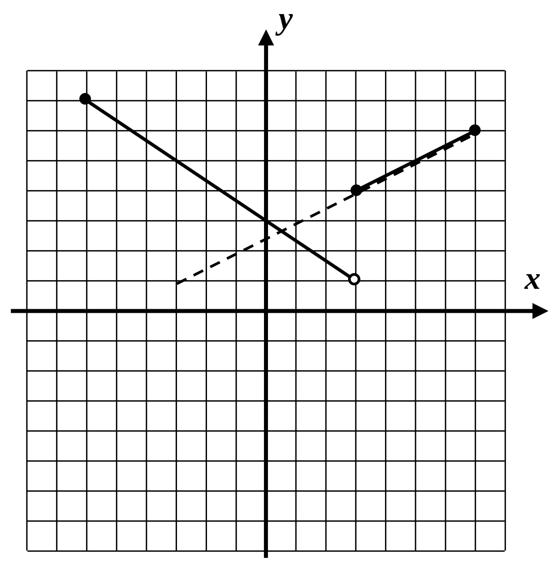
(b) The y-intercept of this function is 8. This is due to the fact that the y-intercept is the output of the function when the input, x, is zero. This was calculated in (iv) above.

(c) $f(-3) = -18; f(9) = 10$
AROC = $\frac{7}{3}$

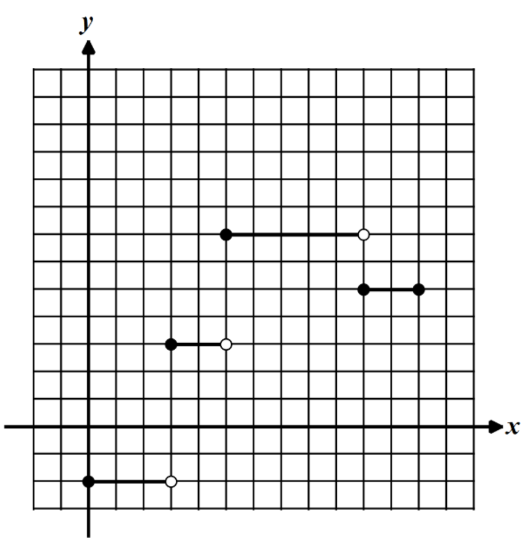
2) Range = $\{0 \le y \le 6\}$



3) $f(x) = \begin{cases} -\frac{2}{3}x + 3 & -6 \le x < 3 \\ \frac{1}{2}x + \frac{5}{2} & 3 \le x \le 7 \end{cases}$



4)



5) $2x + 8 = 0; y - int\colon\ x = -4$

- This is a valid solution because it falls in the interval $-5 \le x < -1$.

$-\frac{1}{2}x - 4 = 0; y - int\colon\ x = -8$

- This is a not a valid solution because it does not fall in the interval $-1 \le x < 1$.

$-4x + 10 = 0; y - int\colon\ x = \frac{5}{2}$

- This is a valid solution because it falls in the interval $1 \le x \le 4$.

LESSON 7
SYSTEMS OF LINEAR FUNCTIONS

Systems of equations, or more than one equation, arise frequently in mathematics. To solve a system means to find all sets of values that simultaneously make all equations true. You have solved systems of linear equations in the last two Common Core math courses, but we will add to their complexity in this lesson.

***Exercise* #1:** Solve the following system of equations by: (a) substitution and (b) by elimination.

(a) $3x + 2y = -9$

$2x + y = -7$

(b) $3x + 2y = -9$

$2x + y = -7$

You should be very familiar with solving two-by-two systems of linear equations (two equations and two unknowns). In this lesson, we will extend the method of **elimination** to linear systems of three equations and three unknowns. These linear systems serve as the basis for a field of math known as **Linear Algebra**.

***Exercise* #2:** Consider the three-by-three system of linear equations shown below. Each equation is numbered in this first exercise to help keep track of our manipulations.

$$
\begin{array}{ll}
(1) & 2x + y + z = 15 \\
(2) & 6x - 3y - z = 35 \\
(3) & -4x + 4y - z = -14
\end{array}
$$

(a) The **addition property of equality** allows us to add two equations together to produce a third valid equation. Create a system by adding equations (1) and (2) and (2) and (3).

(b) Use this new two-by-two system to solve the three-by-three.

In three-by-three systems we may need to use the **multiplication property of equality** before we can eliminate variables.

***Exercise* #3:** Solve the following system of equations. Show all steps.

$4x + y - 3z = -6$
$-2x + 4y + 2z = 38$
$5x - y - 7z = -19$

***Exercise* #4:** Solve the system of equations. Show all steps.

$$4x - 2y + 3z = 23$$
$$x + 5y - 3z = -37$$
$$-2x + y + 4z = 27$$

Lesson 7
Homework

1. The sum of two numbers is 5 and the larger difference of the two numbers is 39. Find the two numbers by setting up a system of two equations with two unknowns and solving algebraically.

2. Algebraically, find the intersection points of the two lines whose equations are shown below.
   $4x + 3y = -13$
   $y = 6x - 8$

3. Show that $x = 10, y = 4, and\ z = 7$ is a solution to the system below *without* solving the system formally.
   $x + 2y + z + 25$
   $4x + 2y + z = 25$
   $-2x - y + 8z = 32$

4. In the following system, the value of the constant c is unknown, but it is known that $x = 8$ and $y = 4$ are the x and y values that solve this system. Determine the value of c. Show how you arrived at your answer.

$$
\begin{aligned}
-5x + 2y + 3z &= 81 \\
x - y + z &= -1 \\
2x - y + cz &= 35
\end{aligned}
$$

5. Solve the following system of equations. Carefully show how you arrived at your answers.

$$
\begin{aligned}
4x + 2y - z &= 21 \\
-x - 2y + 2z &= 13 \\
3x - 2y + 5z &= 70
\end{aligned}
$$

6. Algebraically solve the following system of equations. There are two variables that can be readily eliminated, but your answers will be the same no matter which you eliminate first.

   $2x + 5y - z = -35$
   $x - 3y + 4z = 31$
   $-3x + 2y + 2z = -23$

7. Algebraically solve the following system of equations. This system will take more manipulation because there are no variables with coefficients equal to 1.

   $2x + 3y - 2z = 33$
   $4x + 5y + 3z = 54$
   $-6x - 2y - 8z = -50$

Answers to Lesson 7 – Homework

1) The two numbers are $-17\ and\ 22$.

2) The point of intersection is $\left(\frac{1}{2}, -5\right)$.

3) Prove by substituting the given values into all three equations and prove equality.

4) $z = 11; c = 5$

5) $x = 9, y = -4, z = 7$

6) $x = 3, y = -8, z = 1$

7) $x = 8, y = 5, z = -1$