Chapter 3: Linear Functions, Equations, and their Algebra

Topics:

1 – Direct Variations

2 – Average Rate of Change

3 – Forms of a Line

4 – Linear Modeling

5 – Inverse of Linear Functions

6 – Piecewise Linear Functions

7 – Systems of Linear Equations

LESSON 1
DIRECT VARIATION

We begin our linear unit by looking at the simplest linear relationship that can exist between two variables, namely that of **direct variation**. We say that two variables are **directly related** or **proportional** to one another if the following relationship holds.

PROPORTIONAL OR DIRECT RELATIONSHIPS

Two variables, x and y, have a **direct (proportional) relationship** if for every ordered pair (x, y) we have:

$$\frac{y}{x} = k \text{ or } y = kx$$

Stated succinctly, y will always be a constant multiple of x. The value of k is known as the **constant of variation**.

***Exercise* #1:** In each of the following, x and y are directly related. Solve for the missing value.

(a) y = 15 when x = 5
y = ? when x = 9

(b) y = −6 when x = 4
y = ? when x = −10

(c) y = 12 when x = 16
y = ? when x = 24

***Exercise* #2:** The distance a person can travel varies directly with the time they have been traveling if going at a constant speed. If Phoenix traveled 78 miles in 1.5 hours while going at a constant speed, how far will he travel in 2 hours at the same speed?

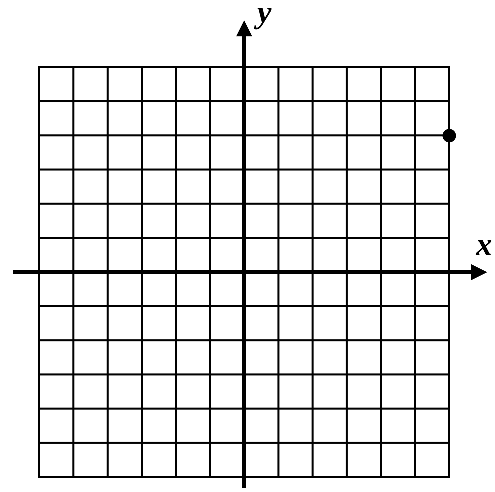
***Exercise* #3:** Jenna works a job where her pay varies directly with the number of hours she has worked. In one week, she worked 35 hours and made $274.75. How many hours would she need to work in order to earn $337.55?

We will now examine the graph of a direct relationship and see why it is indeed the simplest of all linear functions.

***Exercise* #4:** Two variables, *x* and *y*, vary directly. When $x = 6$ then $y = 4$. The point is shown plotted below.

(a) Find the *y*-values for each of the following *x*-values. Plot each point and connect.

$x = 3$ $\qquad\qquad$ $x = -6$



(b) What is the constant of variation in this problem? What does it represent on this line?

(c) Write the equation of the line you plotted in (a).

Direct relationships often exist between two variables whose values are zero simultaneously.

***Exercise* #3:** The miles driven by a car, *d*, varies directly with the number of gallons, *g*, of gasoline used. Abagail is able to drive $d = 336$ miles on $g = 8$ gallons of gasoline in her hybrid vehicle.

(a) Calculate the constant of variation for relationship $\frac{d}{g}$. *Include proper units.*

(b) Give a linear equation that represents the the relationship between *d* and *g*. Express your answer as an equation solved for *d*.

(c) How far can Abagail drive on 6 gallons of gas?

(d) How many gallons of gas will she need in order to drive 483 miles?

LESSON 1
HOMEWORK

1. In each of the following, the variable pair given **are proportional** to one another. Find the missing value.

(a) b = 8 when a = 16
b = ? when a = 18

(b) y = 10 when x = 14
y = ? when x = 21

2. In the following exercises, the two variables given **vary directly** with one another. Find the missing value.

(a) $p = 12$ when $q = 8$
$p = ?$ when $q = 6$

(b) $y = 21$ when $x = 9$
$y = ?$ when $x = -6$

______3. If x and y vary directly and $y = 16$ when $x = 12$, then which of the following equations correctly represents the relationship between x and y?

(1) $y = \frac{3}{4}x$

(2) $y + x = 28$

(3) $xy = 192$

(4) $y = \frac{4}{3}x$

4. The distance Max's bike moves is directly proportional to how many rotations his bike's crank shaft has made. If Max's bike moves 25 feet after two rotations, how many feet will the bike move after 15 rotations?

5. For his workout, the increase in Jacob's heart rate is directly proportional to the amount of time he has spent working out. If his heartbeat has increased by 8 beats per minute after 20 minutes of working out, how much will his heartbeat have increased after 30 minutes of working out?

6. When a photograph is enlarged or shrunken, its width and length stay proportional to the original width and length. Rojas is enlarging a picture whose original width was 3 inches and whose original length was 5 inches. If its new length is to be 8 inches, what is the exact value of its new width in inches?

7. For a set amount of time, the distance Kirk can run is directly related to his average speed. If Kirk can run 3 miles while running at 6 miles per hour, how far can he run in the same amount of time if his speed increases to 10 miles per hour?

8. Two variables are proportional if they can be written at $y = kx$, where k is some constant. This leads to the fact that when $x = 0$ then $y = 0$ as well. Is the temperature measured in Celsius proportional to the temperature measured in Fahrenheit? Explain.

Answer Key to Lesson 1 – Homework

1) (a) b = 9 (b) y = 15

2) (a) p = 9 (b) $y = -14$

3) (4) $y = \frac{4}{3}x$

4) 187.5 feet

5) 12 beats per minute

6) 4.8 inches

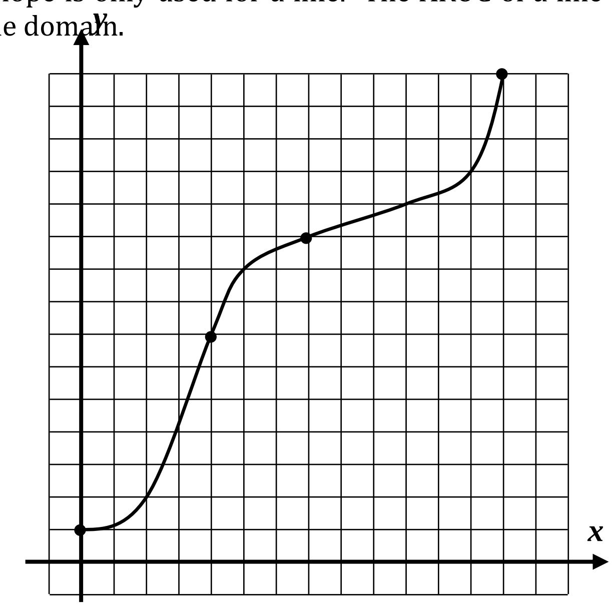
7) 5 miles

8) No, two variables will only be proportional to each other if they are both zero at the same time. Since Celsius temperature is 0 when the Fahrenheit temperature is 32, these two cannot be proportional.

Lesson 2
Average Rate of Change

When we model using functions, we are very often interested in the rate that the output is changing compared to the rate of the input. In Algebra I you were often told to think of the "Slope" when asked to find the Average Rate of Change (AROC). The formula for slope is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. This is very similar to the AROC formula that we will see shortly. The major difference between AROC and slope, is that slope is only used for a line. The AROC of a line is constant, but for other functions the AROC varies depending on the domain.

***Exercise* #1:** The function $f(x)$ is shown graphed to the right.



(a) Evaluate each of the following based on the graph:

(i) $f(0) =$ (ii) $f(4) =$

(iii) $f(7) =$ (iv) $f(13) =$

(b) Calculate the **average rate of change** for the function over each of the intervals and determine which interval has the greatest rate.

(i) $0 \le x \le 4$ (ii) $4 \le x \le 7$ (iii) $7 \le x \le 13$

(c) Using a straightedge, draw in the lines whose slopes you found in part (b) by connecting the points shown on the graph. The average rate of change gives a measurement of what property of the line?

The average rate of change is an exceptionally important concept in mathematics because it gives us a way to **quantify** how fast a function changes on average over a certain **domain interval**. Although we used its formula in the last exercise, we state it formally here:

AVERAGE RATE OF CHANGE

For a function over the domain interval $a \leq x \leq b$, the function's **average rate of change** is calculated by:

$$\frac{\Delta f}{\Delta x} = \frac{change\ in\ the\ output}{change\ in\ the\ input} = \frac{f(b) - f(a)}{b - a}$$

***Exercise* #2:** Consider the two functions $f(x) = 5x + 7$ and $g(x) = 2x^2 + 1$.

(a) Calculate the average rate of change for both functions over the following intervals. Do your work carefully and show the calculations that lead to your answers.

(i) $-2 \leq x \leq 3$

(ii) $1 \leq x \leq 5$

(b) The average rate of change for f was the same for both (i) and (ii) but was not the same for g. Why is that?

***Exercise* #3:** The table below represents a linear function. Fill in the missing entries.

x	1	5	11		45
y	-5	1		22	

LESSON 2
HOMEWORK

1. For the function $g(x)$ given in the table below, calculate the average rate of change for each of the following intervals.

x	-3	-1	4	6	9
$g(x)$	8	-2	13	12	5

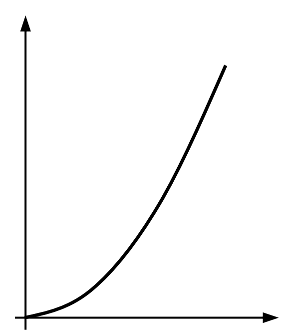
(a) $-3 \leq x \leq -1$ (b) $-1 \leq x \leq 6$ (c) $-3 \leq x \leq 9$

(d) Explain how you can tell from the answers in (a) through (c) that table does **not** represents a linear function.

2. Consider the simple quadratic function $f(x) = x^2$. Calculate the AROC of this function over the following intervals:

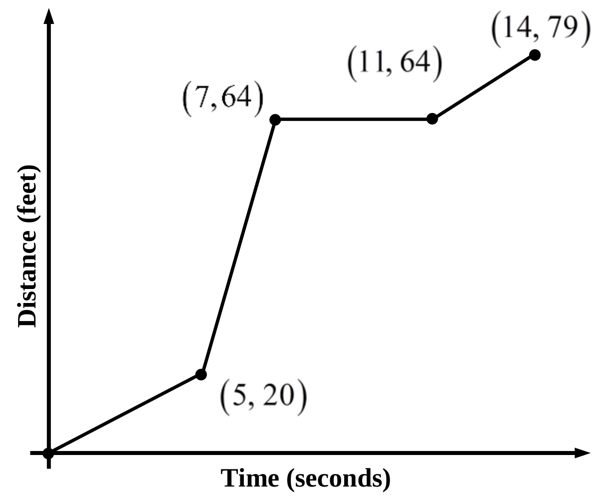
(a) $0 \leq x \leq 2$ (b) $2 \leq x \leq 4$ (c) $4 \leq x \leq 6$

(d) Clearly the average rate of change is getting larger at x gets larger. How is this reflected in the graph of f shown sketched to the right?



3. Which has a greater average rate of change over the interval $-2 \le x \le 4$, the function $g(x) = 16x - 3$ or the function $f(x) = 2x^2$? Provide justification for your answer.

4. An object travels such that its distance, d, away from its starting point is shown as a function of time, t, in seconds, in the graph below.



(a) What is the average rate of change of d over the interval $5 \le t \le 7$? Include proper units in your answer.

(c) The average rate of change of distance over time (what you found in part (a)) is known as the **average speed** of an object. Is the average speed of this object greater on the interval $0 \le t \le 5$ or $11 \le t \le 14$? Justify.

5. What makes the average rate of change of a linear function different from that of any other function? What is the special name that we give to the average rate of change of a linear function?

Answer Key to Lesson 2 – Homework

1) (a) -5 (b) 2 (c) $-\frac{1}{4}$

(d) For linear functions the average rate of change is constant. Since the average rate of change is different for each of these intervals, this is not a linear function.

2) (a) 2 (b) 6 (c) 10

(d) As we move from left to right, the graph is getting steeper.

3) - Since $g(x)$ is a linear function the average rate of change is its slope. The slope for $g(x)$ is 16, which is constant over all intervals.

- The average rate of change for $f(x)$ is 4 (work necessary).

- Therefore, $g(x)$ has a greater average rate of change because 16 is greater than 4.

4) (a) 22 feet per second; or 22 ft/sec

(b) $0 \leq t \leq 5$ = 4 ft/sec;
$11 \leq t \leq 14$= 5 ft/sec

- The average speed is greater in the interval from 11 to 14 because 5 is greater than 4.

5) The average rate of change is a constant for linear functions and does not depend on the interval over which it is calculated. No other type of function behaves this way. We call the average rate of change of a linear function its slope.

LESSON 3
FORMS OF A LINE

Linear functions come in a variety of forms. The following may have been introduced in Alg I and Geo CC.

TWO COMMON FORMS OF A LINE

Slope-Intercept: $y = mx + b$ **Point-Slope:** $y - y_1 = m(x - x_1)$

where m is the slope (or average rate of change) of the line and (x_1, y_1) represents one point on the line.

RECALL: Parallel lines have ______________________.

Perpendicular lines have ___________________________.

***Exercise* #1:** Consider the linear function $f(x) = 3x + 5$

(a) Determine the y-intercept of this function by evaluating $f(0)$.

(b) Find the AROC over the interval $-2 \leq x \leq 3$.

***Exercise* #2:** Consider a line whose slope is 5 and which passes through the point $(-2, 8)$.

(a) Write the equation of this line in point-slope form

(b) Write the equation of this line in slope-intercept form.

____ ***Ex* #3:** Which of the following represents an equation for the line that is parallel to $y = \frac{3}{2}x - 7$ and which passes through the point $(6, -8)$?

(1) $y - 8 = -\frac{2}{3}(x + 6)$ (3) $y + 8 = \frac{2}{3}(x - 6)$

(2) $y - 8 = \frac{3}{2}(x + 6)$ (4) $y + 8 = -\frac{2}{3}(x - 6)$

***Exercise* #4:** A line passes through the points $(5, -2) and\ (20, 4)$.

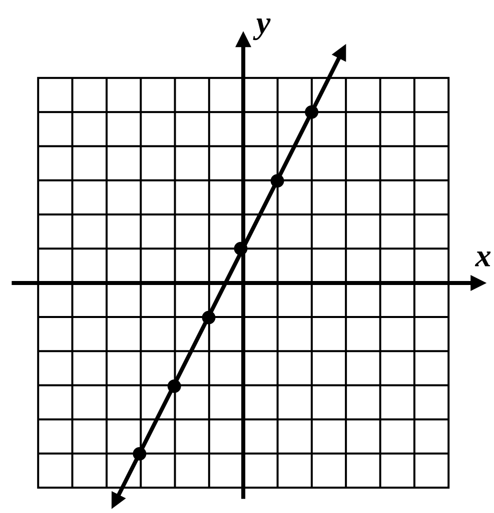
(a) Determine the slope of this line in simplest
slope form.
rational form.

(b) Write the equation of this line in point-

(c) Write an equation for this line in slope-intercept form

(d) For what x-value will this line pass through a y-value of 12?

***Exercise* #5:** The graph of a linear function is shown below.



(a) Write the equation of this line in $y = mx + b$ form.

(b) What must be the slope of a line perpendicular to the one shown?

(c) Draw a line perpendicular to the one shown that passes through the point $(1, 3)$.

(d) Write the equation of the line you just drew in point-slope form.

(e) Does the line that you drew contain the point $(30, -15)$? Justify.

LESSON 3
HOMEWORK

_____1. Which of the following lines is *perpendicular* to $y = \frac{5}{3}x - 7$ and has a y-intercept of 4?

(1) $y = \frac{5}{3}x + 4$ (3) $y = 4x - \frac{3}{5}$

(2) $y = -\frac{3}{5}x + 4$ (4) $y = \frac{3}{5}x + 4$

_____2. Which of the following lines passes through the point $(-4, -8)$?

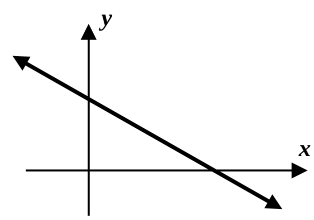
(1) $y + 8 = 3(x + 4)$ (3) $y + 8 = 3(x - 4)$

(2) $y - 8 = 3(x - 4)$ (4) $y - 8 = 3(x + 4)$

_____3. Which of the following equations could describe the graph of the linear function shown below?

(1) $y = \frac{2}{3}x - 4$ (3) $y = -\frac{2}{3}x - 4$

(2) $y = \frac{2}{3}x + 4$ (4) $y = -\frac{2}{3}x + 4$

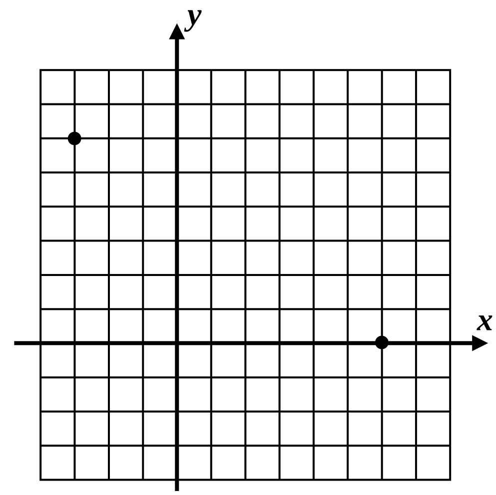


4. For a line whose slope is -3 and which passes through the point $(5, -2)$:

(a) Write the equation of this line in point-slope form.

(b) Write the equation of this line in slope-intercept form.

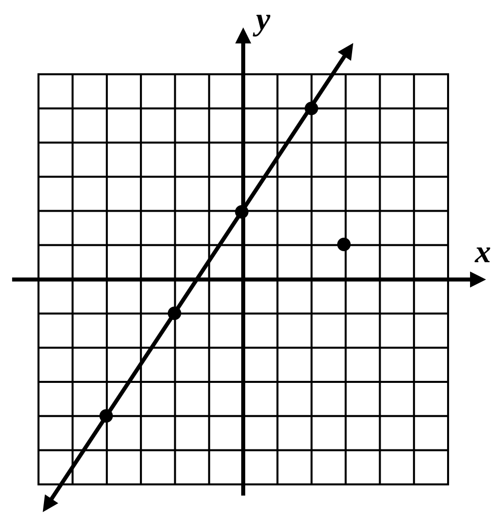
5. The two points $(-3, 6)$ and $(6, 0)$ are plotted on the grid below.



(a) Find an equation, y= $mx + b$ form, for the line passing through these two points. Use of the grid is optional.

(b) Does the point $(30, -16)$ lie on this line? Justify.

6. A linear function is graphed below along with the point $(3, 1)$.



(a) Draw a line parallel to the one shown that passes through the point $(3, 1)$.

(b) Write an equation for the line you just drew in point-slope form.

(c) Between what two consecutive integers does the y-intercept of the line that you drew fall?

(d) Deterimine the *exact* value of the y-intercept of the line that you drew

Answers to Lesson 3 – Homework

1) (2)
2) (1)
3) (4)

4) (a) $y + 2 = -3(x - 5)$

   (b) $y = -3x + 13$

5) (a) $y - 1 = \frac{4}{5}(x + 3)$

   (b) $y = \frac{4}{5}x + 3.4$

6) (a) $y = -\frac{2}{3}x + 4$

   (b) The point does lie on the line because when substituted it makes the equation true.

7) (a) [a line drawn with a slope of $\frac{3}{2}$ that passes through (3, 1).]

   (b) $y - 1 = \frac{3}{2}(x - 3)$

   (c) Between the integers -3 and -4.

   (d) y-intercept = -3.5

LESSON 4
LINEAR MODELING

In Common Core Algebra I, you used linear functions to model any process that had a constant rate at which one variable changes with respect to the other, or a constant slope. In this lesson we will review many of the facets of this type of modeling.

***Exercise* #1:** Dia was driving away from NYC at a constant speed of 58 miles per hour. He started 45 miles away.

(a) Write a linear function that gives Dia's distance, *D*, from NYC as a function of the number of hours, *h*, he has been driving.

(b) If Dia's destination is 270 miles away from NYC, algebraically determine to the nearest tenth of an hour how long it will take Dia to reach his destination.

In *Exercise* #1, it is clear from the context what both the slope and the *y*-intercept of this linear model are. Although this is often the case when constructing a linear model, sometimes the slope and a point are known, in which case, the point slope form of the a line is more appropriate.

***Exercise* #2:** Eve is trying to model her cell-phone plan. She knows that it has a fixed cost, per month, along with a $0.15 charge per call she makes. In her last month's bill, she was charged $12.80 for making 52 calls.

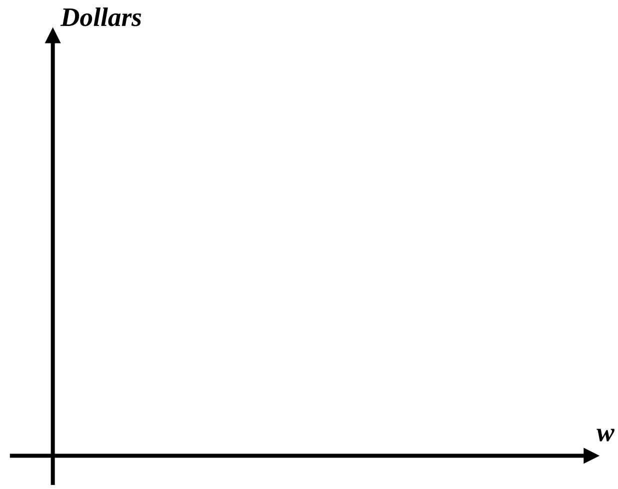
(a) Create a linear model, in point-slope form, for the amount Eve must pay, P, per month given the number of phone calls she makes, c.

(b) How much is Eve's fixed cost? In other words, how much would she have to pay for making zero phone calls?

Many times linear models have been constructed and we are asked only to work with these models. Models in the real world can be messy and it is often convenient to use our graphing calculators to plot and investigate their behavior.

***Exercise* #3:** A factory produces widgets. The cost, *C*, in dollars to produce *w* widgets is given by the equation $C=0.18w+20.64$. Each widget sells for 26 cents. Thus, the revenue gained, *R*, from selling these widgets is given by $R=0.26w$.

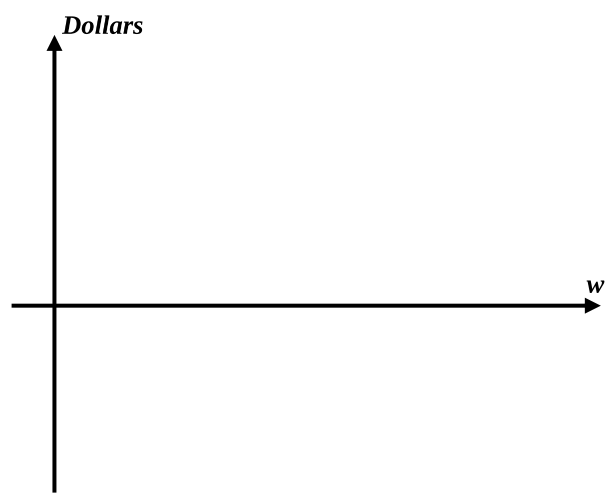
(a) Use your graphing calculator to sketch and label Each of these linear functions for the interval $0 \le w \le 500$. Label the y-axis with its scale.



(b) Use your calculators **INTERSECT** command to determine the number of widgets, *w*, that must be produced for the revenue to equal the cost.

(c) If profit is defined as the revenue minus the cost, create an equation in terms of *w* for the profit, *P.*

(d) Using your graphing calculator, sketch a graph of the profit over the interval $0 \le w \le 1000$. Use a **TABLE** on your calculator to determine an appropriate **WINDOW** for viewing. Label the *x* and *y* intercepts of this line on the graph



(e) What is the minimum number of widgets that must be sold in order for the profit to reach at least $40. Illustrate this on your graph.

Lesson 4
Homework

_____1. Which of the following would model the distance, D, a driver is from Chicago if they are heading *towards* the city at 58 miles per hour and started 256 miles away?

(1) $D = 256t + 58$ (3) $D = 58t + 256$

(2) $D = 256 - 58t$ (4) $D = 58 - 256t$

_____2. The cost, C, of producing x-bikes is given by $C = 22x + 132$. The revenue gained from selling x-bikes is given by $R = 350x$. If the profit, P, is defined as $P = R - C$, then which of the following is an equation for P in terms of x?

(1) $P = 328x - 132$ (3) $P = 328x + 132$

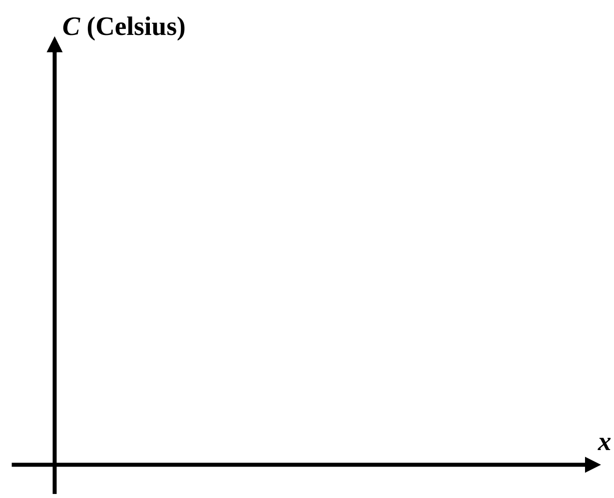
(2) $P = 372x + 132$ (4) $P = 372x - 132$

3. The average temperature of the planet is expected to rise at an average rate of 0.04 degrees Celsius per year due to global warming. The average temperature in the year 2000 was 14.71 degrees Celsius. The average Celsius temperature, C, is given by $C = 14.71 + 0.04x$, where x represents the number of years since 2000.

(a) What will be the average temperature in the year 2100?
degrees

(b) Algebraically determine the number of years, x, it will take for the temperature, C, to reach 20 Celcius. Round to the nearest year.

(c) Sketch a graph of the average yearly temperature below for the interval $0 \le x \le 2000$. Be sure to label your y-axis scale as well as two points on the line. (the y-intercept and on additional point).



(d) What does this model project to be the average global temperature in 2200?

4. Fabio is driving west away from Albany and towards Buffalo along Interstate 90 at a constant rate of speed of 62 miles per hour. After driving for 1.5 hours, Fabio is 221 miles from Albany.

(a) Write a linear model for the distance, D, that Fabio is away from Albany as a function of the number of hours, h, that he has been driving. Write your model in point-slope form, $D - D_1 = m(h - h_1)$.

(b) Rewrite this model in slope-intercept form, $D = mh + b$.

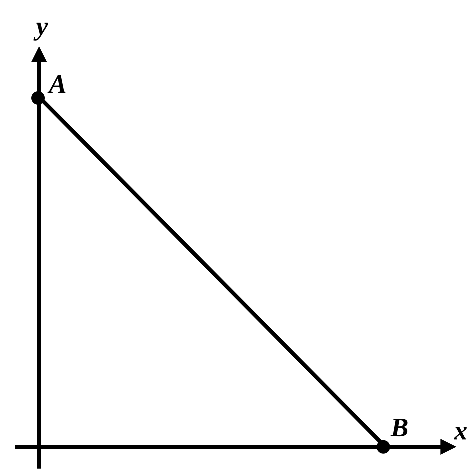
(c) How far was Fabio from Albany when he started his trip?

(d) If the total distance from Albany to Buffalo is 290 miles, determine how long it takes for Fabio to reach Buffalo. Round your answer to the nearest tenth of an hour.

5. A particular rocket taking off from the Earth's surface uses fuel at a constant rate of 12.5 gallons per minute. The rocket initially contains 225 gallons of fuel.

(a) Determine a linear model, in $y = ax + b$ form, for the amount of fuel, y, as a function of the number minutes, x, that the rocket has burned.

(b) Below is a general sketch of what the graph of your model should look like. Using your of calculator, determine the x and y intercepts of this model and label them on the graph at points A and B respectively.

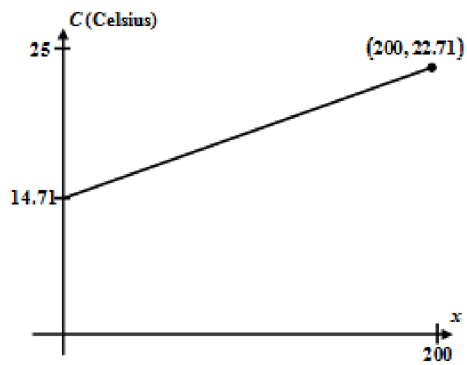


(c) The rocket must still contain 50 gallons of fuel when it hits the stratosphere. What is the maximum number of minutes the rocket can take to hit the stratosphere? Show this point on your graph by also graphing the horizontal line $y = 50$ and showing the intersection point.

Answers to Lesson 4 - Homework

1) (2)
2) (1)

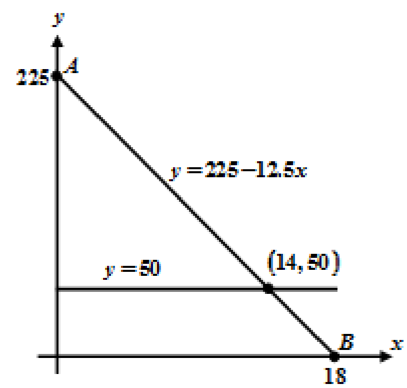
3) (a) $18.71°C$
   (b) approximately 132 years
   (c)



   (d) $22.71°C$

4) (a) $D - 221 = 62(h - 1.5)$
   (b) $D = 62h + 128$
   (c) 128 $miles$; which is the y-intercept of this model
   (d) approximately 2.6 hours

5) (a) $y = -12.5x + 225$
   (b)

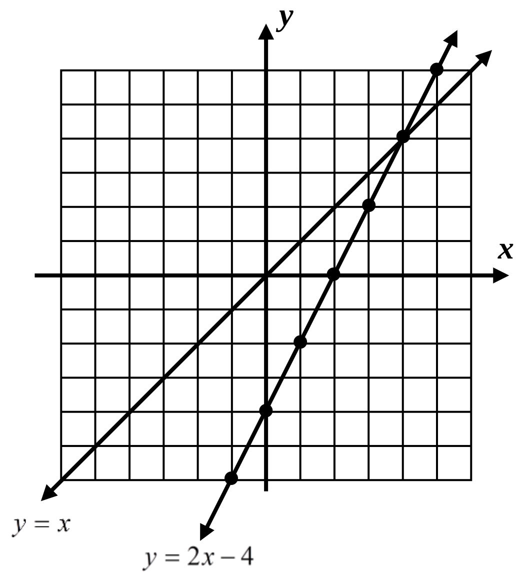


   (c) As shown on the graph, the maximum number of minutes is 14, after that the rocket will contain less than 50 gallons of fuel.

LESSON 5
INVERSE OF LINEAR FUNCTIONS

Recall that functions have inverses that are also functions if they are one-to-one. With the exception of horizontal lines, all linear functions are one-to-one and thus have inverses that are also functions. In this lesson we will investigate these inverses and how to find their equations.

***Exercise* #1:** On the grid below the linear function $y = 2x - 4$ is graphed along with the line $y = x$.



(a) How can you quickly tell that $y = 2x - 4$ is a one-to-one function?

(b) Graph the inverse of $y = 2x - 4$ on the same grid. Recall that this is easily done by switching the x and y coordinates of the original line.

(c) What can be said about the graphs of $y = 2x - 4$ and its inverse with respect to the line $y = x$?

(d) Find the equation of the inverse in . b form.

(e) Find the equation of the inverse in $y = \frac{x+b}{a}$ $y = mx +$ form.

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As we can see from part (e) in *Exercise* #1, inverses of linear functions include the inverse operations of the original function but in reverse order. This gives rise to a simple method of finding the equation of any inverse. **Simply switch the *x* and *y* variables in the original equation and solve for *y*.**

____ ***Exercise* #2:** Which of the following represents the equation of the inverse of $y = 5x - 20$.

(1) $y = -\frac{1}{5}x + 20$ (3) $y = \frac{1}{5}x - 4$

(2) $y = \frac{1}{5}x - 20$ (4) $y = \frac{1}{5}x + 4$

Although this is a simple enough procedure, certain problems can lead to common errors when solving for *y*. Care should be taken with each algebraic step.

____ ***Exercise* #3:** Which of the following represents the inverse of the linear function $y = \frac{2}{3}x + 8$?

(1) $y = \frac{3}{2}x - 8$ (3) $y = -\frac{3}{2}x + 8$

(2) $y = \frac{3}{2}x - 12$ (4) $y = -\frac{3}{2}x + 12$

____ ***Exercise* #4:** What is the *y*-intercept of the inverse of $y = \frac{3}{5}x - 9$?

(1) $y = 15$ (3) $y = 9$

(2) $y = \frac{1}{9}$ (4) $y = -\frac{5}{3}$

Sometimes we are asked to work with linear functions in their point-slope form. The method of finding the inverse and plotting it, though, do not change just because the linear equation is written in a different form.

____***Exercise* #5:** Which of the following would be an equation for the inverse of $y+6=4(x\ -2)$?

(1) $y-2=\frac{1}{4}(x+6)$ (3) $y-6=-4(x+2)$

(2) $y-2=-\frac{1}{4}(x+6)$ (4) $y+2=-4(x-6)$

____***Exercise* #6:** Which of the following points lies on the graph of the inverse of $y-8=5(x+2)$? Explain your choice.

(1) $(8,-2)$ (3) $(-10,40)$

(2) $(-8,2)$ (4) $(-2,8)$

____***Exercise* #7:** Which of the following linear functions would *not* have an inverse that is also a function? Explain how you made your choice.

(1) $y=x$ (3) $y=2$

(2) $2y=x$ (4) $y=5x-1$

LESSON 5
HOMEWORK

_____1. The graph of a function and its inverse are always symmetric across which of the following lines?

(1) $y = 0$ (3) $y = x$

(2) $x = 0$ (4) $y = 1$

_____2. Which of the following represents the inverse of the linear function $y = 3x - 24$?

(1) $y = \frac{1}{3}x + 8$ (3) $y = -\frac{1}{3}x + 24$

(2) $y = -\frac{1}{3}x - 8$ (4) $y = \frac{1}{3}x - \frac{1}{24}$

_____3. If the y-intercept of a linear function is 8, then we know which of the following about its inverse?

(1) Its y-intercept is -8. (3) Its y-intercept is $\frac{1}{8}$.

(2) Its x-intercept is 8. (4) Its x-intercept is -8.

_____4. If both were plotted, which of the following linear functions would be parallel to its inverse? Explain your thinking.

(1) $y = 2x$ (3) $y = 5x - 1$

(2) $y = \frac{2}{3}x - 4$ (4) $y = x + 6$

_____5. Which of the following represents the equation of the inverse of $y = \frac{4}{3x} + 24$?

(1) $y = -\frac{4}{3}x - 24$ (3) $y = \frac{3}{4}x - 18$

(2) $y = -\frac{3}{4}x + 18$ (4) $y = \frac{4}{3}x - 24$

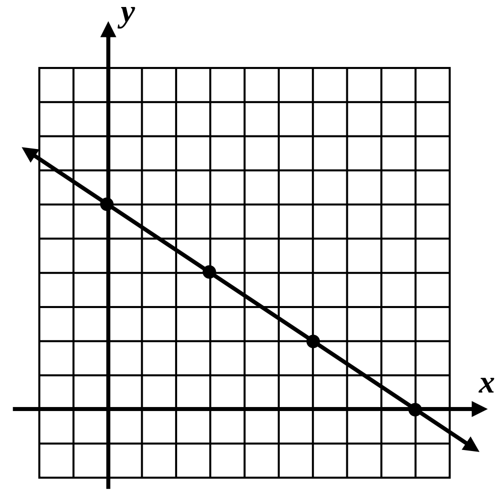
_____6. Which of the following points lies on the inverse of $y + 2 = 4(x - 1)$?

(1) $(2, -1)$ (3) $\left(\frac{1}{2}, 1\right)$

(2) $(-1, 2)$ (4) $(-2, 1)$

7. A linear function is graphed below. Answer the following questions based on this graph.

(a) Write the equation of this linear function in $y = mx + b$ form.



(b) Sketch a graph of the inverse of this function on the same grid.

(c) Write the equation of the inverse in $y = mx + b$ form.

(d) What is the intersection point of this line with its inverse?

8. A car traveling at a constant speed of 58 miles per hour has a distance of y-miles from Poughkeepsie, NY, given by the equation $y = 58x + 24$, where x represents the time in hours that the car has been traveling.

(a) Find the equation of the inverse of this linear function an in $y = \frac{(x-a)}{b}$ form.

(b) Evaluate the function you found in part (a) for input of x = 227.

(c) Give a physical interpretation of the answer you found in part (b). Consider what the input and output of the inverse represent in order to answer this question.

9. Given the general linear function $y = mx + b$, find an equation for its inverse in terms of m and b.

Answers to Lesson 5 - Homework

1) (3)

2) (1)

3) (2)

4) (4)

5) (3)

6) (4)

7) (a) $y = -\frac{2}{3}x + 6$

   (b) Graph a line by switching the x, and y-coordinates from the given line.

   (c) $y = -\frac{3}{2}x + 9$

   (d) $\left(\frac{18}{5}, \frac{18}{5}\right)$

8) (a) $y = \frac{x-24}{58}$ (b) $y = 3.5$

   (c) In the original function x represented the time and y represented the distance. For the inverse, these two must be switched, thus x represents the distance and y represents the time. So, the interpretation of (b) is that it takes 3.5 hours to reach a distance of 227 miles from Poughkeepsie.

9) $y = \frac{1}{m}x - \frac{b}{m}$ or $y = \frac{x-b}{m}$

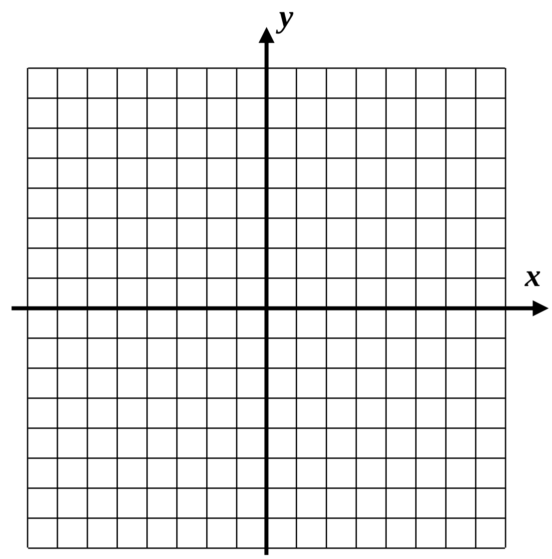
LESSON 6
PIECEWISE LINEAR FUNCTIONS

Functions expressed algebraically can sometimes be more complicated and involve different equations for different portions of their domains. These are known as piecewise functions (they come in pieces). If all of the pieces are linear, then they are known as piecewise linear functions.

Exercise #1: Consider the piecewise linear function given by the formula $\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x-3} & \boldsymbol{-3 \le x < 0} \\ \boldsymbol{\frac{1}{2}x+4} & \boldsymbol{0 \le x \le 4} \end{cases}$

(a) Create a table of values below and graph the function.

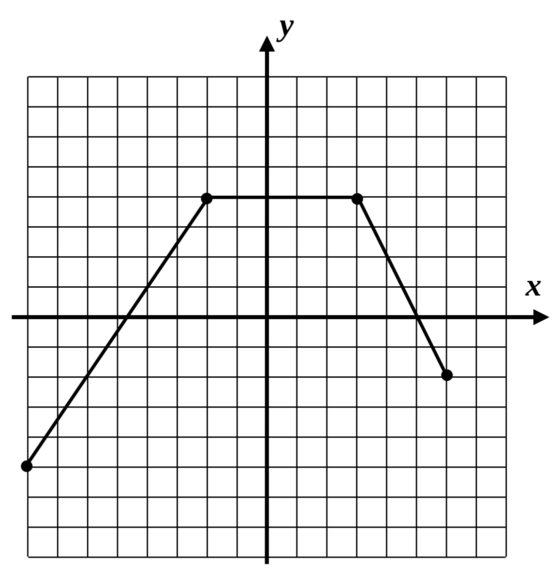
x	–3	–2	–1	0	1	2	3	4
$f(x)$								



(b) State the range of f using interval notation.

Not only should we be able to graph piecewise functions when we are given their equations, but we should also be able to translate the graphs of these functions into equations.

***Exercise* #2:** The function $f(x)$ is shown graphed below. Write a piecewise linear formula for the function. Be sure to specify both the formulas and the domain intervals over which they apply.



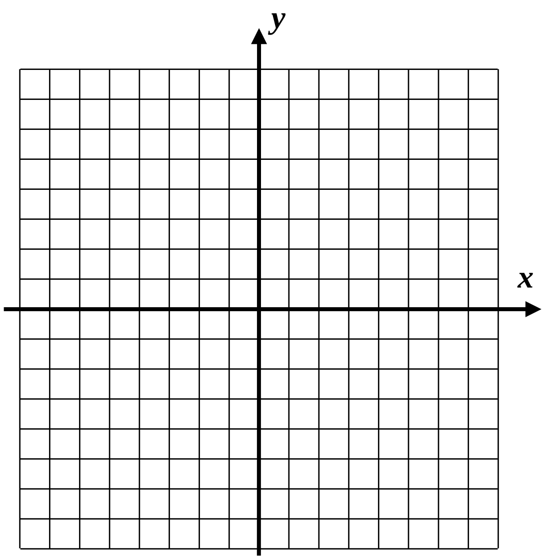
Piecewise equations can be challenging algebraically. Sometimes information that we find from them can be misleading or incorrect.

***Exercise* #3:** Consider the piecewise linear function $g(x) = \begin{cases} 5 - x & x < 2 \\ \frac{1}{2}x + 2 & x \geq 2 \end{cases}$

(a) Determine the y-intercept of this function algebraically. Why can a function have only one y-intercept?

(b) Find the x-intercepts of each individual linear equation.

(c) Graph the piecewise linear function below.



(d) Why does your graph contradict the answers you found in part (b)?

(e) How can you resolve the fact that the algebra seems to contradict your graphical evidence of x-intercepts?

***Exercise* #4:** For the piecewise linear function $f(x) = \begin{cases} -2x + 10 & x \leq 0 \\ 5x - 1 & x > 0 \end{cases}$, find all solutions to the equation $f(x) = 1$ algebraically.

LESSON 6
HOMEWORK

1. For $f(x) = \begin{cases} 5x - 3 & x < -2 \\ x + 8 & -2 \leq x < 3 \\ \frac{1}{3}x + 7 & x \geq 3 \end{cases}$ answer the following questions.

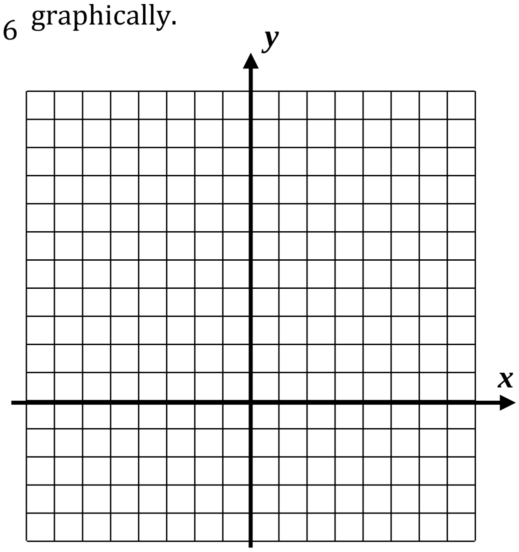
   (a) Evaluate each of the following by carefully applying the correct formula:

   (i) $f(2)$ (ii) $f(-4)$ (iii) $f(3)$ (iv) $f(0)$

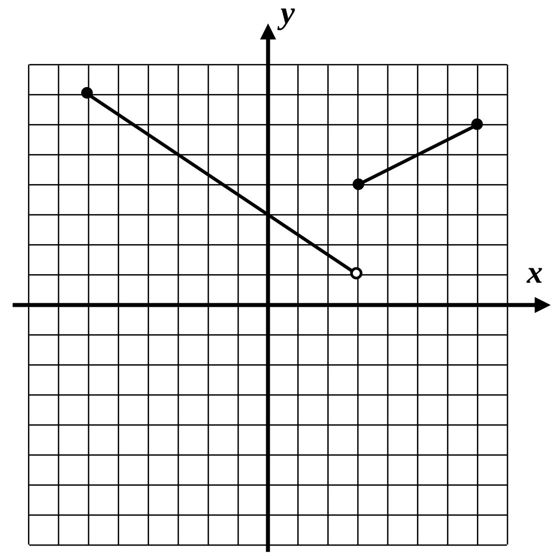
   (b) The three linear equations have y-intercepts of $-3, 8, and\ 7$ respectively. Yet, a function can have only one y-intercept. Which of these is the y-intercept of this function? Explain how you made your choice.

   (c) Calculate the average rate of change of f over the interval $-3 \leq x \leq 9$. Show the calculations that lead to your answer.

2. Determine the range of the function $g(\boldsymbol{x}) = \begin{cases} x + 4 & -2 \leq x < 2 \\ -\frac{3}{2}x + 9 & 2 < x \leq 6 \end{cases}$ graphically.

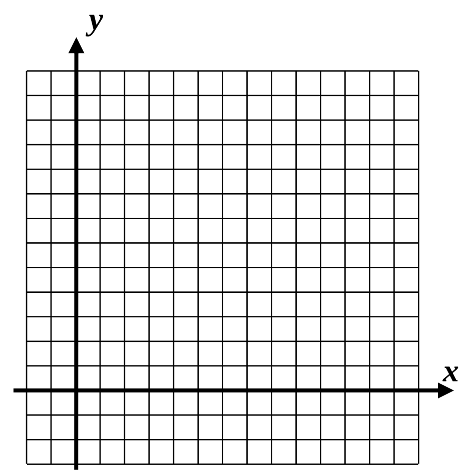


3. Determine a piecewise linear equation for the function $f(x)$ shown below. Be sure to specify not only the equations, but also the domain intervals over which they apply.



4. Step functions are piecewise functions that are constants (horizontal lines) over each part of their domains. Graph the following step function.

$$f(x)\begin{cases} -2 & 0 \le x < 3 \\ 3 & 3 \le x < 5 \\ 7 & 5 \le x < 10 \\ 5 & 10 \le x \le 12 \end{cases}$$



5. Find all x-intercepts of the function $g(x) = \begin{cases} 2x + 8 & -5 \le x < -1 \\ -\frac{1}{2}x - 4 & -1 \le x < 1 \\ -4x + 10 & 1 \le x \le 4 \end{cases}$ algebraically. Justify your work by showing your algebra. Be sure to check your answers versus the domain intervals to make sure each solution is valid.

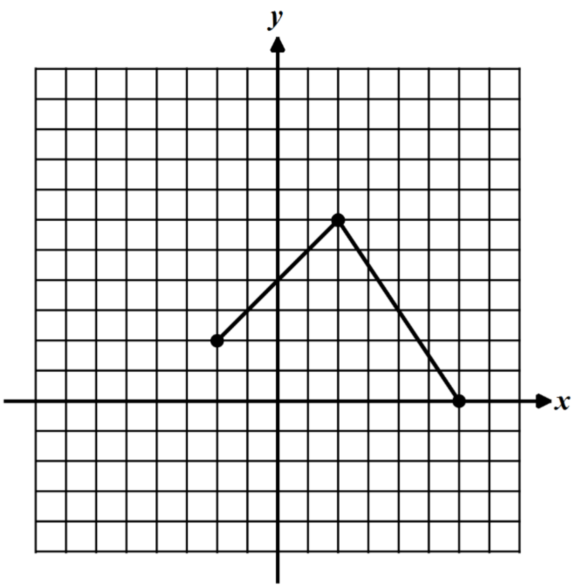
Answers to Lesson 6 – Homework

1) (a) (i) 10 (ii) -23 (iii) 8 (iv) 8

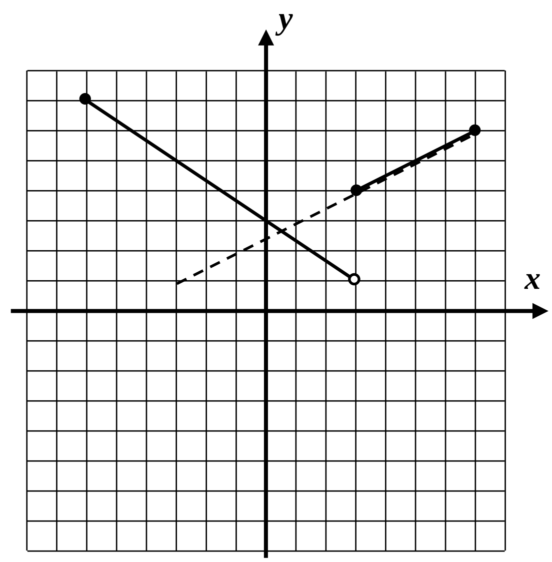
(b) The y-intercept of this function is 8. This is due to the fact that the y-intercept is the output of the function when the input, x, is zero. This was calculated in (iv) above.

(c) $f(-3) = -18; f(9) = 10$
AROC = $\frac{7}{3}$

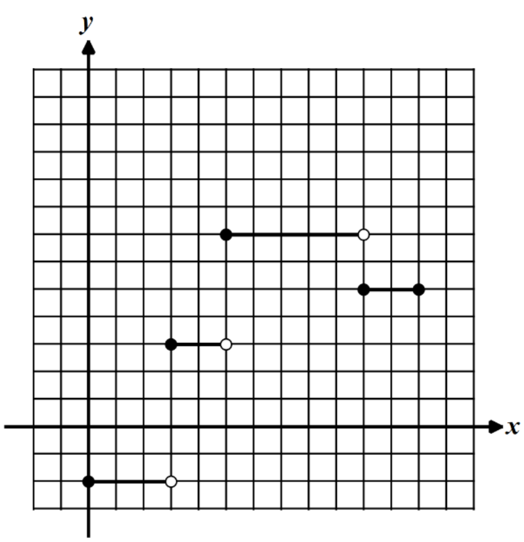
2) Range = $\{0 \leq y \leq 6\}$



3) $f(x) = \begin{cases} -\frac{2}{3}x + 3 & -6 \leq x < 3 \\ \frac{1}{2}x + \frac{5}{2} & 3 \leq x \leq 7 \end{cases}$



4)



5) $2x + 8 = 0; y - int{:}\ x = -4$

- This is a valid solution because it falls in the interval $-5 \leq x < -1$.

$-\frac{1}{2}x - 4 = 0; y - int{:}\ x = -8$

- This is a not a valid solution because it does not fall in the interval $-1 \leq x < 1$.

$-4x + 10 = 0; y - int{:}\ x = \frac{5}{2}$

- This is a valid solution because it falls in the interval $1 \leq x \leq 4$.

LESSON 7
SYSTEMS OF LINEAR FUNCTIONS

Systems of equations, or more than one equation, arise frequently in mathematics. To solve a system means to find all sets of values that simultaneously make all equations true. You have solved systems of linear equations in the last two Common Core math courses, but we will add to their complexity in this lesson.

***Exercise* #1:** Solve the following system of equations by: (a) substitution and (b) by elimination.

(a) $3x + 2y = -9$

$2x + y = -7$

(b) $3x + 2y = -9$

$2x + y = -7$

You should be very familiar with solving two-by-two systems of linear equations (two equations and two unknowns). In this lesson, we will extend the method of **elimination** to linear systems of three equations and three unknowns. These linear systems serve as the basis for a field of math known as **Linear Algebra**.

***Exercise* #2:** Consider the three-by-three system of linear equations shown below. Each equation is numbered in this first exercise to help keep track of our manipulations.

$$\begin{array}{ll} (1) & 2x + y + z = 15 \\ (2) & 6x - 3y - z = 35 \\ (3) & -4x + 4y - z = -14 \end{array}$$

(a) The **addition property of equality** allows us to add two equations together to produce a third valid equation. Create a system by adding equations (1) and (2) and (2) and (3).

(b) Use this new two-by-two system to solve the three-by-three.

In three-by-three systems we may need to use the **multiplication property of equality** before we can eliminate variables.

***Exercise* #3:** Solve the following system of equations. Show all steps.

$4x + y - 3z = -6$
$-2x + 4y + 2z = 38$
$5x - y - 7z = -19$

***Exercise* #4:** Solve the system of equations. Show all steps.

$$4x - 2y + 3z = 23$$
$$x + 5y - 3z = -37$$
$$-2x + y + 4z = 27$$

LESSON 7
HOMEWORK

1. The sum of two numbers is 5 and the larger difference of the two numbers is 39. Find the two numbers by setting up a system of two equations with two unknowns and solving algebraically.

2. Algebraically, find the intersection points of the two lines whose equations are shown below.
   $4x + 3y = -13$
   $y = 6x - 8$

3. Show that $x = 10, y = 4, and\ z = 7$ is a solution to the system below *without* solving the system formally.
   $x + 2y + z + 25$
   $4x + 2y + z = 25$
   $-2x - y + 8z = 32$

4. In the following system, the value of the constant c is unknown, but it is known that $x = 8$ and $y = 4$ are the x and y values that solve this system. Determine the value of c. Show how you arrived at your answer.

$$\begin{aligned} -5x + 2y + 3z &= 81 \\ x - y + z &= -1 \\ 2x - y + cz &= 35 \end{aligned}$$

5. Solve the following system of equations. Carefully show how you arrived at your answers.

$$\begin{aligned} &4x + 2y - z = 21 \\ &-x - 2y + 2z = 13 \\ &3x - 2y + 5z = 70 \end{aligned}$$

6. Algebraically solve the following system of equations. There are two variables that can be readily eliminated, but your answers will be the same no matter which you eliminate first.

   $2x + 5y - z = -35$
   $x - 3y + 4z = 31$
   $-3x + 2y + 2z = -23$

7. Algebraically solve the following system of equations. This system will take more manipulation because there are no variables with coefficients equal to 1.

   $2x + 3y - 2z = 33$
   $4x + 5y + 3z = 54$
   $-6x - 2y - 8z = -50$

Answers to Lesson 7 – Homework

1) The two numbers are $-17 \ and \ 22$.

2) The point of intersection is $\left(\frac{1}{2}, -5\right)$.

3) Prove by substituting the given values into all three equations and prove equality.

4) $z = 11; c = 5$

5) $x = 9, y = -4, z = 7$

6) $x = 3, y = -8, z = 1$

7) $x = 8, y = 5, z = -1$